CS 295: Causal Reasoning

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Identification of Causal Effect
The Back-Door Criterion

Primer chapter 3, Causality 1.3.3.1,3.2)
Outline (chapter 3)

• The semantic of Intervention in SCM, the do operators
• How to determine $P(Y|\text{do}(x))$ given an SCM
• The back door criterion and the adjustment formula
• Identifiability
Target: to Determine the Effect of Interventions

• “Correlation is no causation”, e.g., Increasing ice-cream sales is correlated with more crime, still selling more ice-cream will not cause more violence. Hot weather is a cause for both.

• **Randomized controlled experiments** are used to determine causation: all factors except a selected one of interest are kept static or random. So the outcome can only be influenced by the selected factor.

• Randomized experiments are often not feasible (we cannot randomize the weather), so how can we determine cause for wildfire?

• **Observational studies** must be used. But how we untangle correlation from causation?
Causal Inference — Connecting Different Worlds

Data

\[ P \]
Distribution (Regime 1)

\[ P' \]
Distribution (Regime 2)

\[ Q(P') \]
(Asspects of \( P' \))

Inference

What happens when \( P \) changes?

e.g., Infer whether less people would get cancer if we ban smoking.

\[ Q = P(Cancer = true \mid do(Smoking = no)) \]

Not an aspect of \( P \).
The Challenge of Causal Inference

- **Goal:** how much $Y$ changes with $X$ if we vary $X$ between two different constants free of the influence of $Z$.
- **These variations are called causal effects!**

**Real world**

$$P(z, x, w, y)$$

**Hypothetical world**

$$P(y \mid do(x))$$

$Z$ : age, sex  
$X$ : action  
$W$ : mediator  
$Y$ : outcome
**Method for Computing Causal Effects: Randomized Experiments**

Real world

```
X \rightarrow W \rightarrow Y
```

Hypothetical world

```
X \rightarrow W \rightarrow Y
```

\( Z \): age, sex

\( X \): action

\( W \): mediator

\( Y \): outcome

```
\text{change}
```

Randomization:

- \( P(y \mid do(X_0)) \)
- \( P(y \mid do(X_1)) \)

Often we cannot do this:

How do we force people to smoke (and wait 20 years for them to die or not)

How can we change cholesterol levels...

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Computing Causal Effects (l2) from Observational Data (l1)

Questions:

* What is the relationship between $P(z, x, w, y)$ and $P(y \mid do(x))$?
* Is $P(y \mid do(x)) = P(y \mid x)$?
Causal Effects (formal)

*Causal Effect* (Def. 3.2.1 [C]):

Given two disjoint sets of variables, $X$ and $Y$, the causal effect of $X$ on $Y$, denoted as $P(y \mid do(x))$, is a function from $X$ to the space of probability distributions of $Y$.

For each realization $x$ of $X$, $P(y \mid do(x))$ gives the probability $Y = y$ induced by deleting from the model all equations corresponding to variables in $X$ and substituting $X = x$ in the remaining equations.
Computing Causal Effects from Observational Data

Real world

Alternative world

$Z : \text{age, sex}$

$X : \text{action}$

$W : \text{mediator}$

$Y : \text{outcome}$

$M = \begin{cases} 
Z = f_Z(u_z) \\
X = f_X(z, u_x) \\
W = f_W(x, u_w) \\
Y = f_Y(w, z, u_y) 
\end{cases}$

$M_x = \begin{cases} 
Z = f_Z(u_z) \\
X = f_X(z, u_x) \quad X = x \\
W = f_W(x, u_w) \\
Y = f_Y(w, z, u_y) 
\end{cases}$
Computing Causal Effects from Observational Data

Real world

\[ P(v) = \]

\[ P(z) \times P(x | z) \times P(w | x) \times P(y | w, z) \]

change

Alternative world

\[ P_x(v) = \]

\[ P(z) \times P(x | z) \times P(w | x) \times P(y | w, z) \]

d\( \text{do}(X=x) \)

\( P(z) \times P(x | z) \times \text{equal to 1 in } M_x \)

Z : age, sex

X : action

W : mediator

Y : outcome
Outline (chapter 3)

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Computing Causal Effects from Observational Data

Consider a distribution over the variables: season, sprinkler, rain, wet, and slippery; and the causal graph:

This distribution decomposes as

\[
P(v) = P(se)P(sp \mid se)P(ra \mid se)P(we \mid sp, ra)P(sl \mid we)
\]
Computing Causal Effects from Observational Data

Queries:

\[ Q_1 = P(wet \mid Sprinkler = on) \]

\[ Q_2 = P(wet \mid do(Sprinkler = on)) \]
Computing Causal Effects from Observational Data

Queries:

\[ Q_1 = P(wet \mid Sprinkler = on) = P(p_1) + P(p_2) \]

\[ Q_2 = P(wet \mid do(Sprinkler = on)) \]
Computing Causal Effects from Observational Data

Queries:

\[ Q_1 = P(wet \mid Sprinkler = on) \]

\[ = \frac{\sum_{se, ra} P(we \mid Sp = on, ra)P(Sp = on \mid se)P(ra \mid se)P(se)}{\sum_{se} P(Sp = on \mid se)P(se)} \]

\[ Q_2 = P(wet \mid do(Sprinkler = on)) \]

You can do algorithm bucket elimination to infer Q1.
Computing Causal Effects from Observational Data

Queries:

\[ Q_1 = P(wet \mid Sprinkler = on) \]

\[ Q_2 = P(wet \mid do(Sprinkler = on)) \]

You can do algorithm bucket elimination to infer Q2.
Computing Causal Effects from Observational Data

Queries:

\[ Q_1 = P(wet \mid Sprinkler = on) \]

\[ Q_2 = P(wet \mid do(Sprinkler = on)) = P(p_1) \]

You can do algorithm bucket elimination to infer Q1.
Computing Causal Effects from Observational Data

Queries:

\[ Q_1 = P(\text{wet} \mid \text{Sprinkler} = \text{on}) \]

\[
= \frac{\sum_{se,ra} P(\text{wet} \mid Sp = \text{on}, ra)P(\text{Sp} = \text{on} \mid se)P(\text{ra} \mid se)P(se)}{\sum_{se} P(\text{Sp} = \text{on} \mid se)P(se)}
\]

\[ Q_2 = P(\text{wet} \mid \text{do(Sprinkler} = \text{on})) \]

\[
= \frac{\sum_{se,ra} P(\text{wet} \mid Sp = \text{on}, ra)P(\text{Sp} = \text{on})P(\text{ra} \mid se)P(se)}{P(\text{Sp} = \text{on})} \]

equal to 1

You can do algorithm bucket elimination to infer Q2.
Bucket elimination
Algorithm BE-bel (Dechter 1996)

\[ P(A | E = 0) = \alpha \sum_{E=0,D,C,B} P(A) \cdot P(B | A) \cdot P(C | A) \cdot P(D | A, B) \cdot P(E | B, C) \]

bucket B: \[ P(b|a) \cdot P(d|b,a) \cdot P(e|b,c) \]

bucket C: \[ P(c|a) \cdot \lambda^B(a,d,c,e) \]

bucket D: \[ \lambda^C(a,d,e) \]

bucket E: \[ \lambda^D(a,e) \]

bucket A: \[ P(a) \cdot \lambda^E(a) \]

\[ P(a,e=0) = \frac{P(a,e=0)}{P(e=0)} \]

elimination operator

"induced width" (max clique size)
Corollary (Truncated Factorization, Manipulation Thm., G-comp.):

The distribution generated by an intervention $do(X=x)$ (in a Markovian model $M$) is given by the truncated factorization:

$$P(v \mid do(x)) = \prod_{\{v_i \in V \setminus X\}} P(v_i \mid pa_i) \bigg|_{X=x}$$
Truncated Factorization Formula

The truncated product,

$$P(\mathbf{v} \mid \text{do}(\mathbf{x})) = \prod_{\{v_i \in \mathbf{v} \setminus \mathbf{x}\}} P(v_i \mid p a_i) \bigg|_{X=x}$$

can be rewritten as:

$$P(\mathbf{v} \mid \text{do}(\mathbf{x})) = \frac{P(\mathbf{v})}{P(\mathbf{x} \mid p a_x)} \bigg|_{X=x}$$

Also equivalent to:

$$P(\mathbf{v} \mid \text{do}(\mathbf{x})) = P(\mathbf{v} \mid \mathbf{x}, p a_x) P(p a_x) \bigg|_{X=x}$$

The transformation between the observation and interventional distributions can be seen as a re-weighing process.
Intervention vs. Conditioning, The Ice-Cream Story

When we intervene to fix a value of a variable, we curtail the natural tendencies of the variable to vary in response to other variables in nature.

- This corresponds to a surgery of the model
- i.e. varying Z will not affect X
- intervention is different than conditioning.
- Intervention depends on the structure of the graph.

Conditioning $P(Y=y|X=x)$
Intervening $P(Y=y|\text{do}(X=x))$

Figure 3.1: A graphical model representing the relationship between temperature ($Z$), ice cream sales ($X$), and crime rates ($Y$)

Figure 3.2: A graphical model representing an intervention on the model in Figure 3.1 that lowers ice cream sales
Intervention vs Conditioning, The Surgery Operation

The Simpson story

Conditioning $P(Y=y | X=x)$

The blood pressure story

Intervening $P(Y=y | \text{do}(X=x))$

The ice-cream story

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Intervention vs. Conditioning...

In notation, we distinguish between cases where a variable $X$ takes a value $x$ naturally and cases where we fix $X = x$ by denoting the latter $\text{do}(X = x)$. So $P(Y = y \mid X = x)$ is the probability that $Y = y$ conditional on finding $X = x$, while $P(Y = y \mid \text{do}(X = x))$ is the probability that $Y = y$ when we intervene to make $X = x$. In the distributional terminology, $P(Y = y \mid X = x)$ reflects the population distribution of $Y$ among individuals whose $X$ value is $x$. On the other hand, $P(Y = y \mid \text{do}(X = x))$ represents the population distribution of $Y$ if everyone in the population had their $X$ value fixed at $x$. We similarly write $P(Y = y \mid \text{do}(X = x), Z = z)$ to denote the conditional probability of $Y = y$, given $Z = z$, in the distribution created by the intervention $\text{do}(X = x)$.

Do operation and graph surgery can help determine causal effect

We make an assumption that intervention has no side-effect. Namely, assigning a variable by intervention does not affect other variables in a direct way.
The Adjustment Formula

To find out how effective the drug is in the population, we imagine a hypothetical intervention by which we administer the drug uniformly to the entire population and compare the recovery rate to what would obtain under the complementary intervention, where we prevent everyone from using the drug.

We want to estimate the “causal effect difference,” or “average causal effect” (ACE).

\[ P(Y = 1 | do(X = 1)) - P(Y = 1 | do(X = 0)) \] (3.1)

We need a causal story articulated by a graph (for the Simpson story):
Definition of Intervention and Graph Surgery: The Adjustment Formula

• We simulate the intervention in the form of a graph surgery.
• The causal effect $P(Y = y | do(X = x))$ equals to the conditional probability $P_m(Y = y | X = x)$ that prevails in the manipulated model of the figure below.

Important: the random functions for $Z$ and $Y$ remain invariant.
The Adjustment Formula

\[ P(Y = y|do(X = x)) = P_m(Y = y|X = x) \text{ (by definition)} \quad (3.2) \]

\[ = \sum_z P_m(Y = y|X = x, Z = z)P_m(Z = z|X = x) \quad (3.3) \]

\[ = \sum_z P_m(Y = y|X = x, Z = z)P_m(Z = z) \quad (3.4) \]

Equation (3.3) is obtained from Bayes’ rule by conditioning on and summing over all values of \( Z = z \) (as in Eq. (1.19)), while (Eq. 3.4) makes use of the independence of \( Z \) and \( X \) in the modified model.

Finally, using the invariance relations, we obtain a formula for the causal effect, in terms of preintervention probabilities:

\[ P(Y = y|do(X = x)) = \sum_z P(Y = y|X = x, Z = z)P(Z = z) \quad (3.5) \]

Equation (3.5) is called the *adjustment formula* and as you can see, it computes the association between \( X \) and \( Y \) for each value \( z \) of \( Z \), then averages over those values. This procedure is referred to as “adjusting for \( Z \)” or “controlling for \( Z \).”
The Adjustment Formula (in the Simpson story)

\[ P(Y = y|do(X = x)) = \sum_z P(Y = y|X = x, Z = z)P(Z = z) \]  \hspace{1cm} (3.5)

The right hand-side can be estimated from the data since it has only conditional probabilities.

If we had a randomized controlled experiments on X (taking the drug) we would not need adjustment because the data is already generated from the manipulated distribution. Namely it will yield \( P(Y=y|do(x)) \) from the data of the randomized experiment.

In practice adjustment is sometime used in randomized experiments to reduce sampling variations (Cox 1958). (This means: If the input is sampled from the intervened upon joint distribution over X,Y and Z we can estimate the \( P(y|x) \) directly. Or, we can first estimate \( P(y|x,s) \) and also \( P(z) \) and perform the summation.)
In the Simpson example:

We get that the Average Causal Effect (ACE):

\[
P(Y = 1|do(X = 1)) = P(Y = 1|X = 1, Z = 1)P(Z = 1) + P(Y = 1|X = 1, Z = 0)P(Z = 0)
\]

Substituting the figures given in Table 1.1 we obtain

\[
P(Y = 1|do(X = 1)) = \frac{0.93(87 + 270)}{700} + \frac{0.73(263 + 80)}{700} = 0.832
\]

while, similarly,

\[
P(Y = 1|do(X = 0)) = \frac{0.87(87 + 270)}{700} + \frac{0.69(263 + 80)}{700} = 0.7818
\]

We get that the Average Causal Effect (ACE):

\[
ACE = P(Y = 1|do(X = 1)) - P(Y = 1|do(X = 0)) = 0.832 - 0.7818 = 0.0502
\]

A more informal interpretation of ACE is that it is the difference in the fraction of the population that would recover if everyone took the drug compared to when no one takes the drug.
The Blood Pressure Example

Figure 3.5: A graphical model representing the effects of a new drug, with $X$ representing drug usage, $Y$ representing recovery, and $Z$ representing blood pressure (measured at the end of the study). Exogenous variables are not shown in the graph, implying that they are mutually independent.

$$P(Y = y \mid do(X = x)) = ?$$

Here the “surgery on $X$ changes nothing. So,

This means that no surgery is required; the conditions under which data were obtained were such that treatment was assigned “as if randomized.” If there was a factor that would make subjects prefer or reject treatment, such a factor should show up in the model; the absence of such a factor gives us the license to treat $X$ as a randomized treatment.

$$P(Y = y \mid do(X = x)) = P(Y = y \mid X = x),$$

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To Adjust or not to Adjust?

**Rule 1 (The Causal Effect Rule)** Given a graph \( G \) in which a set of variables \( PA \) are designated as the parents of \( X \), the causal effect of \( X \) on \( Y \) is given by

\[
P(Y = y | do(X = x)) = \sum_{z} P(Y = y | X = x, PA = z)P(PA = z)
\]

Where \( z \) ranges over all the combinations of values that the variables that \( PA \) take.

So, the causal graph helps determine the parents \( PA \)!

But, in many cases some of the parents are unobserved so we cannot perform the calculation.

Luckily we can often adjust for other variables substituting for the unmeasured variables in \( PA(X) \), and this Can be decided via the graph.
Multiple Interventions, the Truncated Product Rule

Often we have multiple interventions that may not correspond to disconnected variables. We will use the product decomposition. We write the **product truncated formula**

\[
P(x_1, x_2, \ldots, x_n|do(x)) = \prod_i P(x_i|pa_i) \quad \text{for all } i \text{ with } X_i \text{ not in } X.
\]

Example:

\[
P(z_1, z_2, w, y|do(T = t, Z_3 = z_3)) = P(z_1)P(z_2)P(w|t)P(y|w, z_3, z_2)
\]

where we have deleted the factors \(P(t|z_1, z_3)\) and \(P(z_3|z_1, z_2)\) from the product.
Multiple Interventions and the Truncated Product Rule

preintervention distribution in the model of Figure 3.3 is given by

$$P(x,y,z) = P(z)P(x|z)P(y|x,z)$$

whereas the postintervention distribution, governed by the model of Figure 3.4 is given by the product

$$P(z,y|do(x)) = P_m(z)P_m(y|x,z) = P(z)P(y|x,z) \quad (3.9)$$

with the factor $P(x|z)$ purged from the product, since $X$ becomes parentless as it is fixed at $X = x$. This coincides with the adjustment formula, because to evaluate $P(y|do(x))$ we need to marginalize (or sum) over $z$, which gives

$$P(y|do(x)) = \sum_z P(z)P(y|x,z)$$
Outline (chapter 3)

• The semantic of Intervention in SCM, the do operators
• How to determine $P(Y|\text{do}(x))$ given an SCM
• The identification problem
• The back door criterion and the adjustment formula
The Identification Problem

Causal Effect Identifiability (Def. 3.2.2)

The causal effect of $X$ on $Y$ is said to be identifiable from a causal diagram $G$ if the quantity $P(y \mid do(x))$ can be computed uniquely from a positive probability of the observed variables.

That is, if for every pair of models $M_1$ and $M_2$ inducing $G$, $P_{M_1}(y \mid do(x)) = P_{M_2}(y \mid do(x))$, whenever $P_{M_1}(v) = P_{M_2}(v) > 0$. 
The Identification Problem (II)

Causal Inference

For any two SCMs $M_1, M_2$, $G = G(M_1) = G(M_2)$

$P(v \mid do(x))$ (Unobserved (output))

$(P_{M1}(v) = P_{M2}(v))$ (Observed (input))

$(P_{M1}(y \mid do(x)) = P_{M2}(y \mid do(x)))$
The Identification Problem (II)

For any two SCMs $M_1, M_2$, causal inference is identifiable.
The Identification Problem (II)

Identifiability really means that, no matter the shape of $M_1, M_2$, for all models agreeing in terms of $\langle G, P(v) \rangle$, they will also agree in $P(v \mid do(x))$!
Example. Identifiable Effect

• Consider any two pair of models compatible with the following graph and the same observational distribution \( P(v) \):

\[
P(v) = P(z)P(x|z)P(y|x,z)
\]

\[
P(v|do(x)) = P(z)P(y|x,z) \implies P(y | do(x)) = \sum z P(z)P(y | x,z)
\]

No matter what the specific functions or \( P(u) \) are, as long as \( M_1, M_2 \) agree in \( \langle G, P(v) \rangle \), they will also agree in \( P(z) \) and \( P(y|x,z) \), hence in \( P(v | do(x)) \)!
Example. Non-identifiable Effect

• Consider the pair of models compatible with the following graph $G$ and observational distribution $P(v)$:

$$P(v) = \sum_{u_{xy}} P(y|x, u_{xy}) P(x|u_{xy}) P(u_{xy})$$

$$M^{(1)} = \begin{cases} X &\leftarrow U_{xy} \\ Y &\leftarrow (X \oplus U_{xy}) \lor U_y \end{cases}$$

$$M^{(2)} = \begin{cases} X &\leftarrow U_{xy} \\ Y &\leftarrow U_y \end{cases}$$

$$P^{(j)}(U_i = 1) = 1/2, \quad i = \{x, y, xy\}, j = \{1,2\}$$

They match in $P(v)$, that is, $P^{(1)}(v) = P^{(2)}(v)$!
Example. Non-identifiable Effect

- Consider the pair of models compatible with the following graph $G$ and observational distribution $P(v)$:

$$P(v) = \sum_{u_{xy}} P(y|x, u_{xy}) P(x|u_{xy}) P(u_{xy})$$

$P(v|do(x)) = \sum_{u_{xy}} P(y|x, u_{xy}) P(u_{xy})$

$$M^{(1)} = \begin{cases} X \leftarrow U_{xy} \\ Y \leftarrow (X \oplus U_{xy}) \lor U_y \end{cases}$$

$$M^{(2)} = \begin{cases} X \leftarrow U_{xy} \\ Y \leftarrow U_y \end{cases}$$

Even though both models induce $G$ and have the same $P(v)$, the effect $P^{(1)}(y|do(x)) \neq P^{(2)}(y|do(x))$!
Let’s study how to **decide** whether a causal effect is **identifiable**…
Identification in Markovian Models

Theorem. Given the causal diagram $G$ of any Markovian model that all variables are measured, the causal effect $Q = P(y \mid do(x))$ is identifiable for every subsets of variables $X$ and $Y$ and is obtained from the truncated factorization, i.e.,

$$P(v \mid do(x)) = \prod_{V_i \in V \setminus X} P(v_i \mid pa_i)$$

Sum over all variables not in $X \cup Y$

$$P(y \mid do(x)) = \sum_{v \setminus (x \cup y)} \prod_{V_i \in V \setminus X} P(v_i \mid pa_i)$$
Adjustment by Direct Parents

Thm. Given a causal diagram $G$ of any Markovian system, the causal quantity $Q = P(y \mid do(x))$ is identifiable whenever \( \{X, Y, Pa_x\} \subseteq V \), that is, whenever $X$, $Y$, and all the parents of variables $X$ are measured. The expression of $Q$ is then obtained by adjustment for $PA_x$, or

$$P(y \mid do(x)) = \sum_{pa_x} P(y \mid x, pa_x) P(pa_x)$$

Quiz: 1) derive from previous slide
2) derive for non-Markovian models
How could adjustment help in real data analysis?

(The Problem of Confounding)
Confounding Bias

What’s the causal effect of Exercise on Cholesterol?
What about $P(\text{cholesterol} \mid \text{exercise})$?

Increase exercise $\rightarrow$ increase cholesterol?
Confounding Bias

What’s the causal effect of Exercise on Cholesterol?
What about $P(\text{cholesterol} \mid \text{exercise})$?

More exercise $\rightarrow$ Lower cholesterol (per age group)

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What’s the causal effect of Exercise on Cholesterol?

What about \( P(\text{cholesterol} \mid \text{exercise}) \)?

\[
P(\text{cholesterol} \mid \text{exercise}) \neq P(\text{cholesterol} \mid \text{do}(\text{exercise}))
\]

This difference is called **confounding bias** and represents one of the major obstacles to causal inference & interpretability.

More exercise → Lower cholesterol (per age group)

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If Season is latent, is the effect still computable?

Queries:

\[ Q_2 = P(\text{wet} \mid \text{do}(\text{Sprinkler} = \text{on})) \]

\[ = \frac{\sum_{se,ra} P(we \mid Sp = \text{on}, ra) P(Sp = \text{on}) P(ra \mid se) P(se)}{P(Sp = \text{on})} \]

\[ = \sum_{se,ra} P(we \mid Sp = \text{on}, ra) P(ra \mid se) P(se) \]

\[ = \sum_{se,ra} \sum_{ra} P(we \mid Sp = \text{on}, ra) P(ra, se) \]

\[ = \sum_{ra} P(we \mid Sp = \text{on}, ra) \sum_{se} P(ra, se) \]

\[ = \sum_{ra} P(we \mid Sp = \text{on}, ra) P(ra) \]

Adjustment by Rain!
If Season is latent, is the effect still computable?

Queries:

\[ Q_2 = P(wet \mid do(Sprinkler = on)) \]

\[ = \sum_{ra} P(wet \mid Sp = on, ra)P(ra) \]

By conditioning on rain,
- \( p_2 \) (the non-causal path) is blocked, and
- \( p_1 \) (the causal path) remains unaffected!
Is Confounding Bias removable?

**Goal**: Find the effect of $X$ on $Y$, $Q = P(y|do(x))$, given measurements on variables $Z_1,..., Z_k$, where some of $X$ parents are unobserved.

How can the target quantity $Q$ be identified if only a subset of the parents is measured?
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3.3 The Backdoor Criterion

Definition 3.3.1 (The Backdoor Criterion) Given an ordered pair of variables \((X, Y)\) in a directed acyclic graph \(G\), a set of variables \(Z\) satisfies the backdoor criterion relative to \((X, Y)\) if no node in \(Z\) is a descendant of \(X\), and \(Z\) blocks every path between \(X\) and \(Y\) that contains an arrow into \(X\).

If a set of variables \(Z\) satisfies the backdoor criterion for \(X\) and \(Y\), then the causal effect of \(X\) on \(Y\) is given by the formula

\[
P(Y = y|do(X = x)) = \sum_z P(Y = y|X = x, Z = z)P(Z = z)
\]

Rationale:
1. We block all spurious paths between \(X\) and \(Y\).
2. We leave all directed paths from \(X\) to \(Y\) unperturbed.
3. We create no new spurious paths.
The Back-door Adjustment

Theorem 3.3.2 (Back-door Adjustment)

If a set $Z$ satisfies the bdc w.r.t the pair $X,Y$, the effect of $X$ on $Y$ is identifiable and given by:

$$P(y \mid do(x)) = \sum_z P(y \mid x, z)P(z)$$
Back-Door Sets as Substitutes of the Direct Parents of $X$

Rain satisfies the back-door criterion relative to Sprinkler and Wet:

(i) Rain is not a descendant of Sprinkler, and
(ii) Rain blocks the only back-door path from Sprinkler to Wet.

Adjusting for the direct parents of Sprinkler, we have:

\[
P(we | do(sp)) = \sum_{se} P(we | sp, se)P(se)
= \sum_{se, ra} P(we | sp, se, ra)P(ra | sp, se)P(se)
= \sum_{se, ra} P(we | sp, ra)P(ra | se)P(se)
= \sum_{ra, se} P(we | sp, ra) \sum_{se} P(ra, se) = \sum_{ra} P(we | sp, ra)P(ra)
\]

Direct derivation, showing it works

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Adjustment by Direct Parents \rightarrow Back-door Adjustment

More Generally:
(i) no node in \( Z \) is a descendent of \( X \); and
(ii) \( Z \) blocks every path between \( X \) and \( Y \) that contains an arrow into \( X \).

Then:
\[
P(y \mid do(x)) = \sum_{pa_x} P(y \mid x, pa_x)P(pa_x)
\]
\[
= \sum_{z, pa_x} P(y \mid x, pa_x, z)P(z \mid x, pa_x)P(pa_x)
\]
\[
= \sum_{z, pa_x} P(y \mid x, z)P(z \mid pa_x)P(pa_x)
\]
\[
= \sum_{z} P(y \mid x, z) \sum_{pa_x} P(z, pa_x) = \sum_{z} P(y \mid x, z)P(z)
\]

Adjustment by \( Z \) is equivalent to adjustment by direct parents whenever \( Z \) is bd-admissible!
How do we find these bd-sets?

Graphical Condition

\[ P(y \mid do(x)) \] is identifiable if there is a set \( Z \) that \( d \)-separates \( X \) from \( Y \) in \( G_x \) (the graph \( G \) where all arrows emanating from \( X \) are removed.)

\[
P(y \mid do(x)) = \sum_{z_1, z_4} P(y \mid x, z_1, z_4)P(z_1, z_4)
\]
End of 5th class 2022
Back-door Examples

Are there admissible back-door sets (relative to $X, Y$) for the following graphs?
Back-door Examples

Are there admissible back-door sets (relative to $X, Y$) for the following graphs?

\[ Z = \{Z_4, Z_2\}, \{Z_4, Z_5\}, \{Z_4, Z_2, Z_5\} \]

\[ Z = \emptyset \]
Recaping: The Backdoor Criterion

Under what conditions does a causal story permit us to compute the causal effect of one variable on another, from data obtained by passive observations, with no interventions? Since we have decided to represent causal stories with graphs, the question becomes a graph-theoretical problem: Under what conditions is the structure of the causal graph sufficient for computing a causal effect from a given data set?
More Examples for Backdoors

Figure 3.6: A graphical model representing the relationship between a new drug ($X$), recovery ($Y$), weight ($W$), and an unmeasured variable $Z$ (socioeconomic status)

$W$ is a backdoor. Therefore we can compute:

$$P(Y = y|\text{do}(X = x)) = \sum_w P(Y = y|X = x, W = w)P(W = w)$$
Examples

\[ P(Y | \text{do}(X))? \]

No backdoors between X and Y and therefore: \( P(Y | \text{do}(X)) = P(Y | X) \)

What if we adjust for W? ... wrong!!!

But what if we want to determine \( P(Y | \text{do}(X), w) \)? What do we do with the spurious path \( X \rightarrow W \leftarrow Z \leftarrow T \rightarrow Y \)?

If we condition on \( T \), we would block the spurious path \( X \rightarrow W \leftarrow Z \leftarrow T \rightarrow Y \). We can compute:

\[
P(Y = y | \text{do}(X = x), W = w) = \sum_t P(Y = y | X = x, W = w, T = t) P(T = t | W = w)
\]

Example: W can be post-treatment pain
Adjusting for Colliders?

Figure 3.7: A graphical model in which the backdoor criterion requires that we condition on a collider (Z) in order to ascertain the effect of X on Y.

There are 4 backdoor paths. We must adjust for Z, and one of E or A or both.