EQUIVALENCE AND SYNTHESIS OF CAUSAL MODELS

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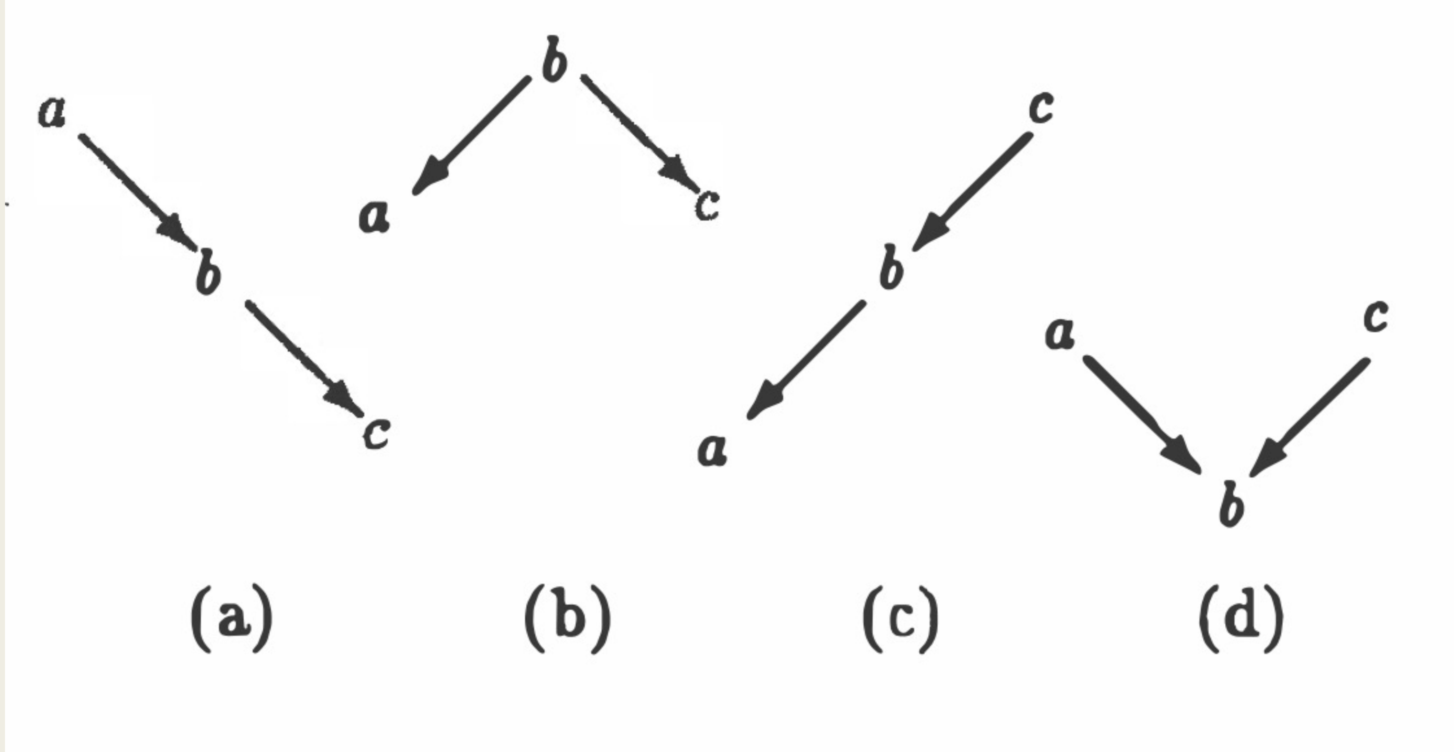
Introduction

- Causal Theory ***T* = <*G*, *P(v)*>**
  - ***G*** is a DAG (causal model)
  - ***P(v)*** is a set of parameters compatible with ***G***
- Equivalence of causal models
  - ***Two causal models G_1 and G_2 are equivalent if for every T_1, there is a T_2 such that both theories describe the same probability distribution***

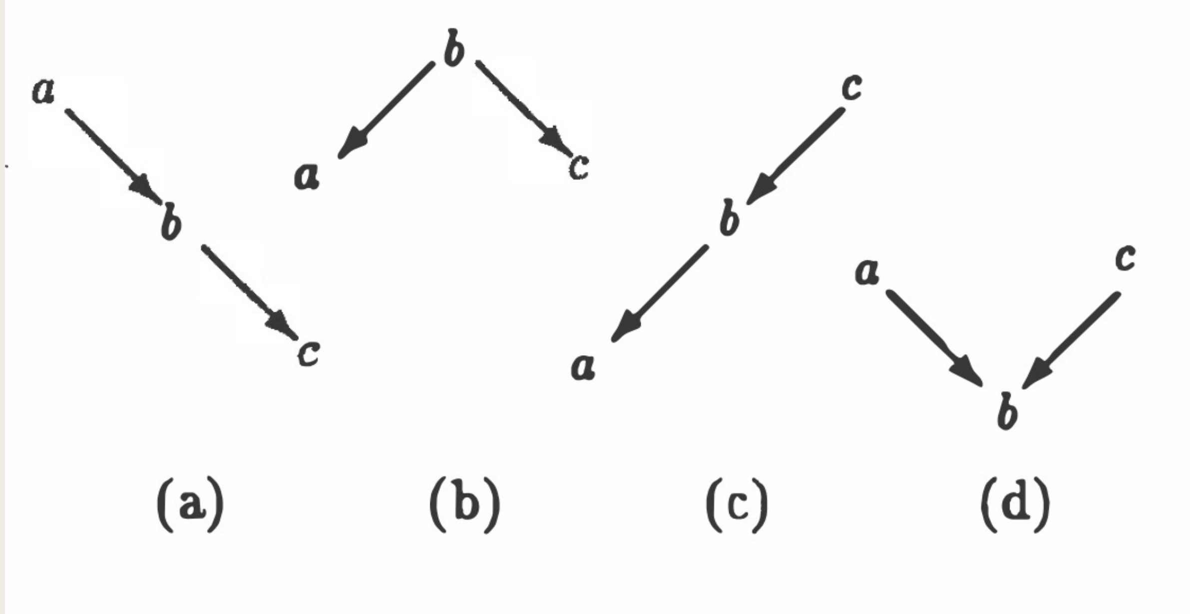
Equivalence – Probability Distribution



A: P(a, b, c) = P(a) P(b|a) P(c|b)
B: P(a, b, c) = P(b) P(a|b) P(c|b)
C: P(a, b, c) = P(c) P(b|c) P(a|b)
D: P(a, b, c) = P(a) P(c) P(b|a, c)

a = b = c $\neq d$

Equivalence – Independence



A, B, C: I (a, b, c)

D: I (a, Ø, c)

Equivalence – Independence

- We can describe any causal model using stratified protocol: a list of independence statements
  - Determine in linear time
- The question of equivalence of causal models is reduced to the question of equivalence protocols
  - ***two DAGs are equivalent if and only if each DAG's protocol holds in the other.***
- Drawbacks
  - ***Does not generalize to embedded causal models***

Outline

- ***Patterns*** for all equivalent models
- Embedded Causal Models
- Applications to Synthesis of Causal Models
- Homework Question

Patterns of Causal Models

- Symbols
  - $\overline{ab}$ are two nodes that are adjacent in a DAG
  - A_{ab} is the set of ancestors of ***a*** and ***b*** in DAG
  - P_{ab} is the set of parents of ***a*** and ***b*** in DAG

Patterns of Causal Models

- **Lemma 1**
- Let ***a*** and ***b*** be two nodes of a DAG ***G***, then the following are equivalent:
  - ***a*** and ***b*** are adjacent in ***G***
  - ***a*** and ***b*** are unseparable in ***G***
  - ***a*** and ***b*** are not d-separated by A_{ab} in ***G***
  - ***a*** and ***b*** are not d-separated by P_{ab} in ***G***

- Adjacency is a property determined solely by d-separation, hence remains invariant among equivalent DAGs

Patterns of Causal Models

- Equivalent DAGs also possess the same directionality of uncoupled colliding links
  - $a \rightarrow b \leftarrow c$ is ***uncoupled*** if ***a*** and ***c*** are not adjacent

- **Lemma 2**

- If the nodes ***a***, ***c***, ***b*** form a chain $\overline{acb}$ where ***a*** and ***b*** are not adjacent, then ***c*** is head-to-head between ***a*** and ***b*** if and only if ***a*** and ***b*** are not separable by any set containing ***c***. That is, for any DAG ***G***, $\overline{acb} \in G$ and $\overline{ab} \notin G$, we have

$$a \rightarrow \mathrm{c} \leftarrow b \in G \iff \neg I(a, \{S, c\}, b) \forall S \subseteq V - abc$$

Patterns of Causal Models

- **Theorem 1**
- Two DAGs are equivalent if and only if they have the same links and same uncoupled head-to-head nodes.
- We can determine the equivalence of two causal models by a simple graphical criterion

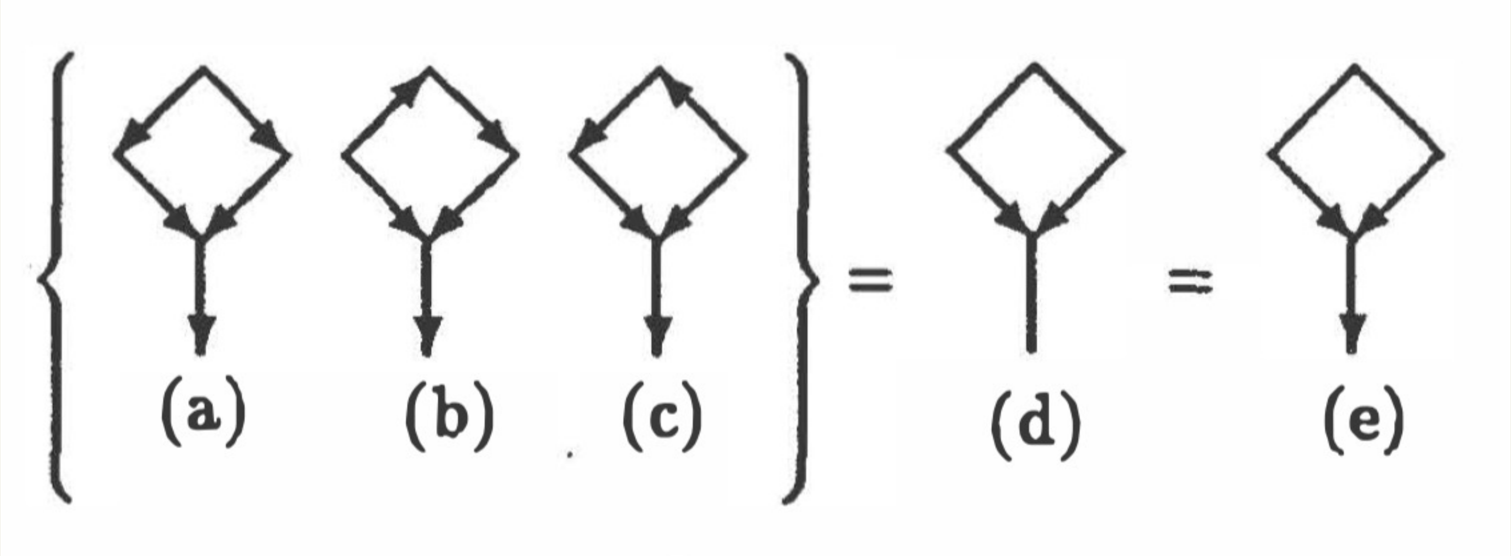
Patterns of Causal Models

- ***Rudimentary pattern*** of causal model
  - Partially directed graph
  - Remove the arrowheads from any link that is not identified by ***Lemma 2***

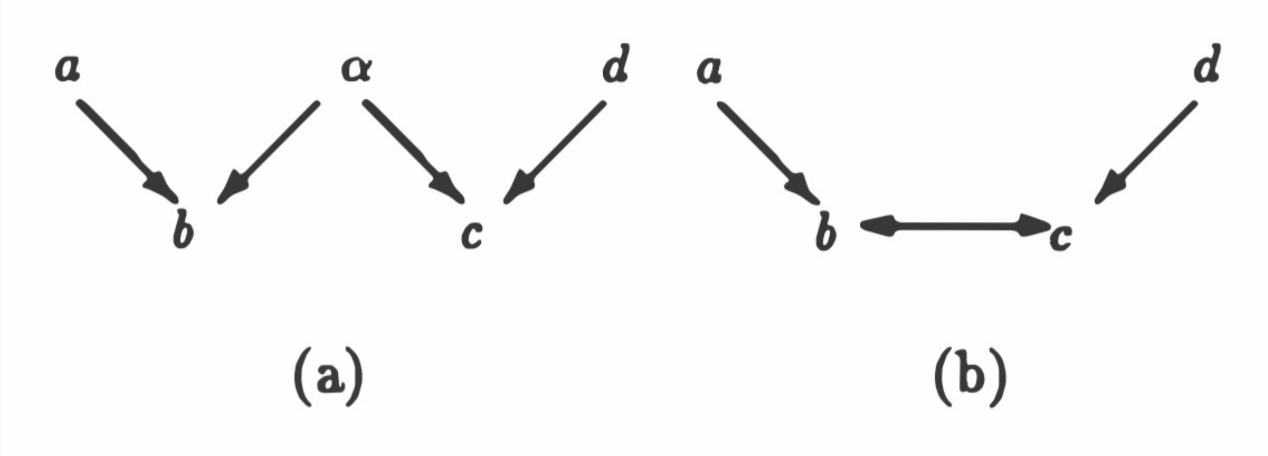
- Rudimentary pattern can be defined solely in terms of d-separation
- Each equivalence class of causal models has a unique pattern

Patterns of Causal Models

- Define ***complete pattern*** that reflects each and every invariant arrowhead



Embedded Causal Models



- Hybrid graph: include bi-directional link

$\underset{ab}{\rightarrow}$ denotes the link has at least an arrowhead pointing at ***b***

$\overline{ab}$ denotes the existence of any link (four types)

Embedded Causal Models

- **Definition 1 (Embedded Pattern)**
- Given a DAG ***G*** of which ***V*** are observable, the rudimentary pattern ***P*** of ***G*** restricted to ***V*** is defined as the hybrid graph with fewest arrowheads that satisfies:
  - $\overline{ab} \in P \Leftrightarrow \neg I(a, S, b)\ \forall S \subseteq V - ab$
  - $\underset{ab}{\rightarrow}$ $if\ \exists\ c \in V\ s.t.\ \overline{abc} \in P, \overline{ac} \notin P\ and\ \neg I(a, \{S, b\}, c)\ \forall S \subseteq V - abc$

- Completed embedded patterns can be derived
- Might require exponential number of tests for d-separation conditions on all subsets of ***V***

Embedded Causal Models

- **Definition 2 (Inducing Path)**
- An inducing path between the variables ***a*** and ***b*** of an embedded causal model is any path ρ satisfying
  - Every observable node on ρ is head-to-head on ρ
  - Every head-to-head node on ρ is in $\mathrm{A_{ab}}$

- **Lemma 3**
- Let ***P*** be the pattern of a DAG ***G*** with respect to ***V*** and $a, b \in V$, the following are equivalent:
  - ***a*** and ***b*** are adjacent in ***P***
  - ***a*** and ***b*** are unseparable in ***G*** over ***V***
  - ***a*** and ***b*** are not d-separated by $A_{ab} \cap V$ in ***G***
  - ***a*** and ***b*** are connected by an inducing path in ***G***

Embedded Causal Models

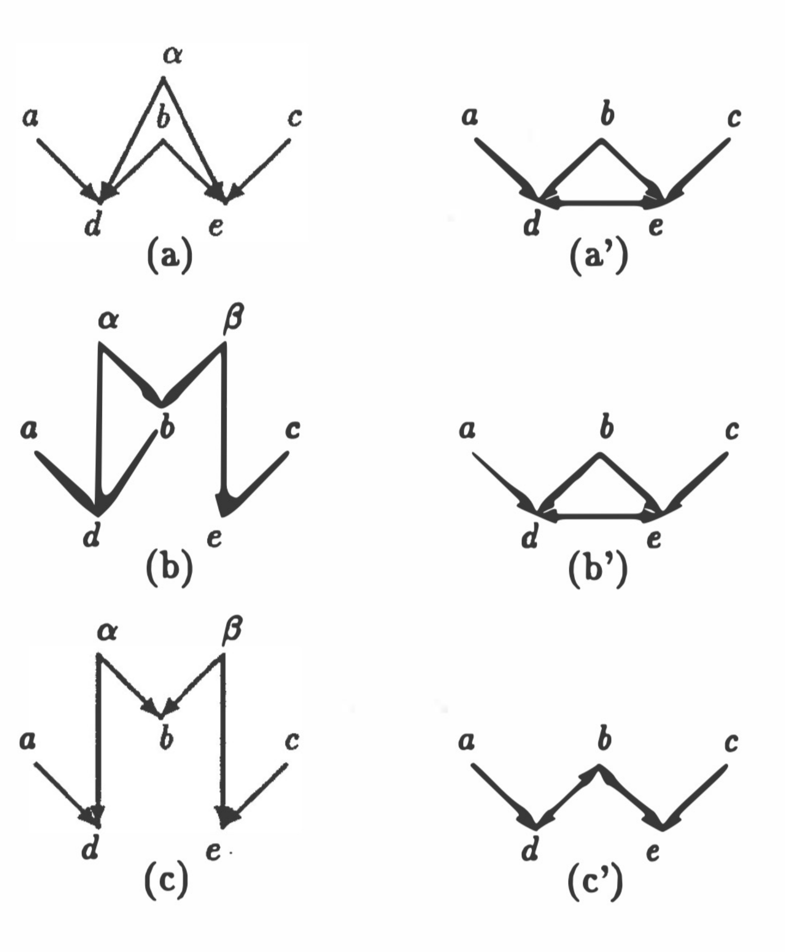
- **Lemma 4**
- For any pattern ***P***, $\overrightarrow{ab}$ if and only if there is a node ***c*** adjacent to ***b*** but not to ***a*** (in ***P***) such that both edges $\overline{ab}$ and $\overline{bc}$ were induced by paths (of ***G)*** which ended pointing at ***b***

- **Theorem 2**
- Two embedded causal models are equivalent if and only if they have the same pattern

Embedded Causal Models

- **Corollary 1**
- There are fewer than $5^{|V|^2}$ distinct embedded causal models containing **|*V*|** variables; moreover, every embedded causal model is equivalent to a simple DAG with fewer than $|V|^2$ variables

Embedded Causal Models



Left side are embedded causal models
Right side are their completed patterns

***V* = *abcde*, *U* =** $\alpha\beta$

(a) and (b) are equivalent, but not to (c)

a and ***b*** are marginally independent in (c), not in (a) and (b)

Application to synthesis of Causal Models

- Possible to extract causal models directly from statistical information
- Difficulty: Probability distributions do not define unique graphical models
  - It is always possible to contrive the parameters to yield spurious independencies, not in ***G***, that fit another causal theory not equivalent to ***G***
- From Spirets et al 1990: Under some assumptions, the occurrence of such spurious independencies is a rare event and it is natural to assume that the underlying distribution is DAG-isomorphic

Recovery Algorithm

- For each pair of variables ***a*** and ***b***, search for a separating set S_{ab} (i.e., such that $I(a, S_{ab}, b)$ holds). If there is no such S_{ab}, place an undirected link between the variables.
- For each pair of non-adjacent variables ***a*** and ***b*** with a common neighbor ***c***, test the statement $I(a, \{c, S_{ab}\}, b)$. If true, continue; otherwise, add arrowheads at ***c***.
- Complete the pattern

- First step dominates complexity: exponential search
  - Adopt Markov network
  - Markov network of a DAG-isomorphic distribution has the property that the parents of any variable in the DAG form a clique in the network
  - **Lemma 1**: separable = separated by parent set
  - Bounded exponentially by the size of the largest clique

Recovery Algorithm

- Drawback of the Markov network reduction
  - ***Not applicable to embedded causal models***
- Use **Theorem 2** to recover
  - ***At most one arrowhead in DAG isomorphic***
  - ***Possible bi-directional link for isomorphic distribution to embedded DAG***
- Invariant nature of arrows in a pattern form the basis for causation

Causation

- **Definition 3 (Genuine and Potential Cause)**
- ***c*** is a genuine cause of ***e*** if ***c*** causes ***e*** in every consistent model (i.e. every pattern contains the directed arrow $c \rightarrow e$). ***c*** is a potential cause of ***e*** if ***c*** causes ***e*** in some consistent model (i.e. some pattern contains $c \rightarrow e$) and ***e*** never causes c in any consistent model (i.e. no pattern contains $c \leftarrow e$) .

- If the distribution is isomorphic to an embedded DAG, then we can find unique pattern by recovery algorithm. Not for general distributions
- From Spirets et al 1990: proposed an algorithm for identifying causal relationships which accepts many of the genuine and potential causes in distributions that are isomorphic to embedded DAGs. Such relationships correspond to the singly directed arrows of the rudimentary pattern.

Causation

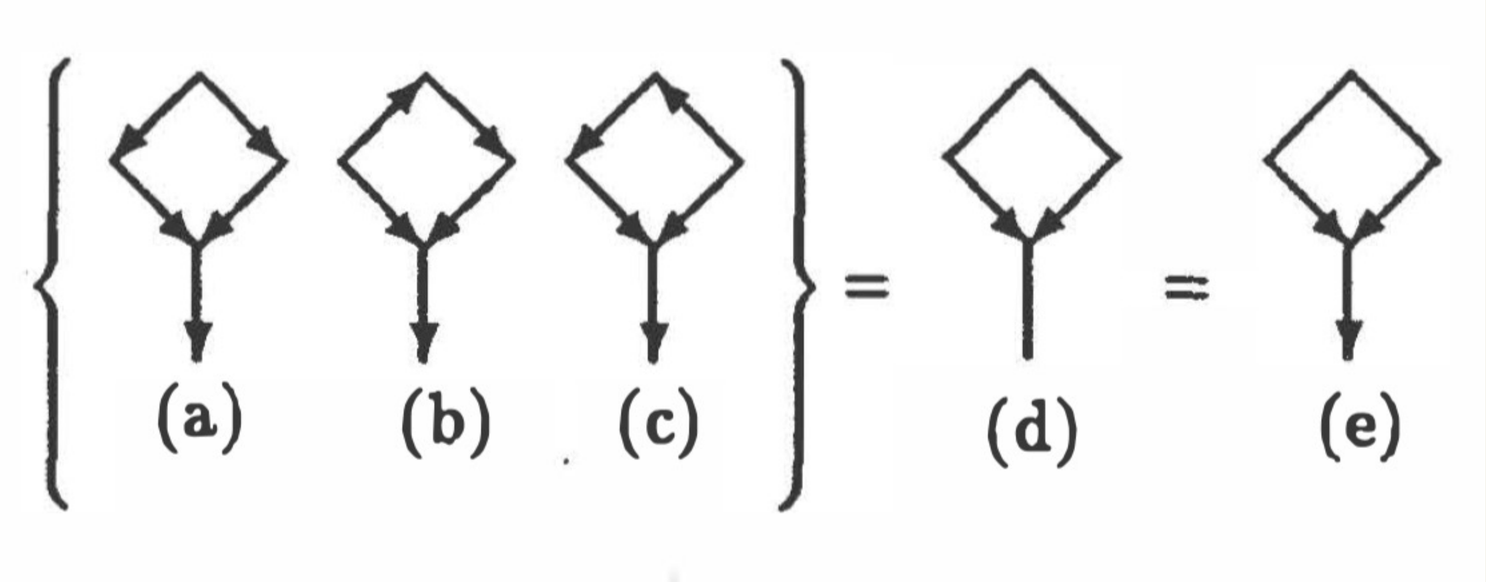
- The number of sampled data required to reliably test the independence statement grows exponentially

- **Definition 4 (Reliable independence)**
- $I(a, S, b)$ holds reliably whenever the set of hypotheses $\{P(a\ |S) = P(a\ |S, b\}$ is confirmed for each instantiation of ***S*** for which a sufficient number of samples are available to reliably test the hypothesis.

- Deterministic node cannot be represented by the causal models, require refinement of d-separation

Homework Question

- Identify the genuine and potential cause in the following pattern if any



References

- Pearl, J. and Verma, T.S. (1990) Equivalence and Synthesis of Causal Models. Proceedings of the 6th Conference on Uncertainty in Artificial Intelligence, Cambridge, 27-29 July, 220-227.
- Spirtes, Peter, Clark Glymour, and Richard Scheines. "Causality from probability." (1989).

Thank You