

Instrumental Variables with Treatment-Induced Selection: Exact Bias Results

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Instrumental Variables

Instrumental variables (IV) analysis is a popular approach for identifying causal effects when the treatment is confounded by omitted variables.

Two Assumptions

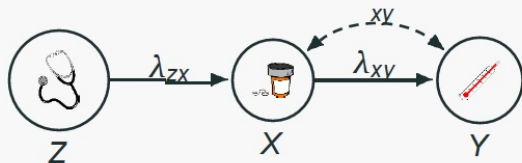
IV analysis rests on two main assumptions:

1. The instrument is associated with the treatment (“relevance”),
2. The instrument is associated with the outcome only via the effect of treatment on the outcome (“exclusion”).

Examples

The Starting Point: Instrumental Variables

Suppose that an independent doctor's recommendation was added to the original drug/biomarker dataset, which influences how much drug patients take.



$$Z := E_Z$$

$$X := \lambda_{zx}Z + E_x$$

$$Y := \lambda_{xy}X + E_y$$

E_x, E_y correlated

$$\sigma_{zx} = \lambda_{zx}$$

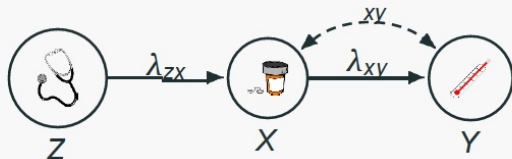
$$\sigma_{zy} = \lambda_{zx}\lambda_{xy}$$

$$\lambda_{xy} = \frac{\sigma_{zy}}{\sigma_{zx}}$$

Examples

The Starting Point: Instrumental Variables

Suppose that an independent doctor's recommendation was added to the original drug/biomarker dataset, which influences how much drug patients take.



A variable Z is an IV (p. 248 [\[Causality\]](#)) for λ_{xy} from X to Y if

- Z is d-separated from Y in the subgraph $G_{\lambda_{xy}}$,
- Z is not d-separated from X in $G_{\lambda_{xy}}$

Causal Graph

A causal graph represents the structural equations of the data-generating model. Causal graphs consist of nodes representing variables and directed edges representing direct causal effects. Causal graphs are assumed explicitly to display the observed and unobserved common causes of all variables. By convention, causal graphs do not explicitly display the idiosyncratic shocks that affect individual variables.

Causal Graph - Cont

Throughout, the authors assume that the causal graphs represent linear data-generating models with homogeneous effects and normally distributed errors. Without loss of generality, the authors further assume that all variables are standardized to have mean zero and unit variance. The direct causal effect of one variable on another variable in such models is given by its path parameter, which is bounded by $[-1,1]$.

Causal Graph

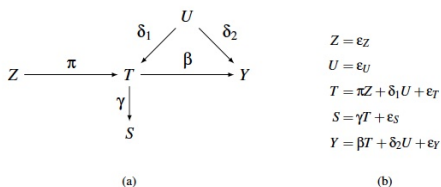
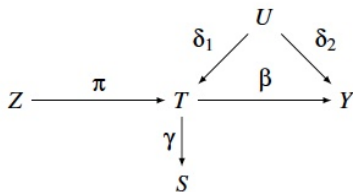


Figure: Figure 1

For example, the causal graph in a represents the linear structural equations model given in b, with path parameters π , β , γ , δ_1 , and δ_2 . For each variable $V \in \{Z, U, T, S, Y\}$ the idiosyncratic shocks are marginally independent and normally distributed, $\varepsilon_V \sim N(0, \delta_V^2)$, with variance δ_V^2 scaled so that each $V \sim N(0, 1)$.

Causal Graph



(a)

$$Z = \varepsilon_Z$$

$$U = \varepsilon_U$$

$$T = \pi Z + \delta_1 U + \varepsilon_T$$

$$S = \gamma T + \varepsilon_S$$

$$Y = \beta T + \delta_2 U + \varepsilon_Y$$

(b)

Since U is unobserved, the structural error term on Y in econometric terminology is $\omega_Y = \delta_2 U + \varepsilon_Y$. Notice that T is correlated with the structural error, $\text{Cov}(T, \omega_Y) \neq 0$, because both depend on the unobserved variable, U .

Instrumental Variables: Def 1

Let T be the treatment variable of interest, Y be the outcome, Z be the candidate instrumental variable, and X be a set of covariates. Economically, an instrumental variable is defined by two assumptions.

Definition 1. A variable, Z , is called an instrumental variable for the causal effect of T on Y , β , if, conditional on the set of covariates X (which may be empty),

E1: Z is associated with T , $Cov(Z, T|X) \neq 0$, (Relevance)

E2: Z is not associated with the structural error term, ω_Y , on Y , $Cov(Z, \omega_Y|X) = 0$. (Exclusion)

Instrumental Variables: Def 2

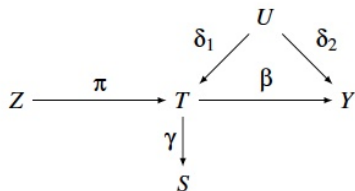
Definition 2. A variable, Z , is called an instrumental variable for the causal effect of T on Y , β , if, conditional on the set of covariates X (which may be empty),

G1: There is at least one open path from Z to T conditional on X ,
(Relevance)

G2: X does not contain descendants of Y , $X \cap \text{desc}(Y) = \emptyset$,
(Exclusion)

G3: There is no open path from Z to Y conditional on X , other than those paths that terminate in a causal path from T to Y .
(Exclusion)

Example



(a)

$$Z = \epsilon_Z$$

$$U = \epsilon_U$$

$$T = \pi Z + \delta_1 U + \epsilon_T$$

$$S = \gamma T + \epsilon_S$$

$$Y = \beta T + \delta_2 U + \epsilon_Y$$

(b)

In a, Z is a valid instrument without conditioning on S , since Z is relevant (associated with T) by the open path $Z \rightarrow T$, and Z is excluded (unassociated with the structural error term on Y) since the only open path from Z to Y , $Z \rightarrow T \rightarrow Y$, terminates in the causal effect of T on Y .

When Z is a valid instrumental variable, then the standard IV estimator, given by the sample analog of:

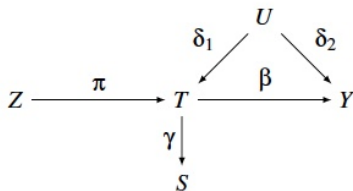
$$\beta_{IV} = \frac{\text{Cov}(Y, Z|X)}{\text{Cov}(T, Z|X)}$$

is consistent for the causal effect of T on Y in linear and homogeneous models. The numerator of this estimator is called the reduced form and the denominator is called the first stage. The behavior of this IV estimator is the focus of the paper. For simplicity, the authors used β_{IV} and β_{OLS} to refer to the probability limits (as the sample size tends to infinity) of the standard IV and OLS estimators, respectively.

Selection Bias in IV: Qualitative Analysis

We say that the IV estimator suffers selection bias when conditioning on some variable violates the exclusion assumption. For example, conditioning on a variable that opens a path between Z and Y that does not terminate in the causal effect of T on Y violates exclusion both in the sense of G3 and E2.

Example



(a)

$$Z = \varepsilon_Z$$

$$U = \varepsilon_U$$

$$T = \pi Z + \delta_1 U + \varepsilon_T$$

$$S = \gamma T + \varepsilon_S$$

$$Y = \beta T + \delta_2 U + \varepsilon_Y$$

(b)

In a, conditioning on S invalidates the use of Z as an instrumental variable, because T is the only collider variable on the path $Z \rightarrow T \leftarrow U \rightarrow Y$, and conditioning on S as the descendant of the collider T opens this path. The association “transmitted” by this open path overtly violates the exclusion condition G3 and similarly violates the exclusion condition E2, since ω_γ is a function of U.

Selection Bias in IV: Qualitative Analysis:Cont

Since conditioning on a variable can result from many different procedures during data collection or data analysis, selection bias in IV analysis can result from many different procedures as well. Analysts should be aware, however, that different ways of conditioning on a variable may induce quantitatively different selection biases.

Selection Bias in IV: Quantitative Analysis

Sources of selection bias

- ▶ **Truncation:** Sample restriction based on values of S . $R = \mathbb{1}(S \geq s_0)$. $(\beta_{IV|Tr}, \beta_{OLS|Tr})$.
- ▶ **Covariate Adjustment:** Controlling for variable S in regression. $(\beta_{IV|Adj}, \beta_{OLS|Adj})$.

Selection Bias in IV: Quantitative Analysis

Selection bias with adjustment

Proposition 1

In a linear and homogeneous model with normal errors represented by Figure 1a and covariate adjustment on S , the standard instrumental variables estimator converges in probability to

$$\beta_{IV|Adj} = \beta - \delta_1 \delta_2 \frac{\gamma^2}{1 - \gamma^2}.$$

Selection bias with adjustment

- ▶ Conditioning on S opens the path $Z \rightarrow T \leftarrow U \rightarrow Y$.
- ▶ When $\gamma = 0$, bias is 0. No relationship.
- ▶ Bias unbounded as $|\gamma| \rightarrow 1$. Adjusting for S same as adjusting for collider T .

Selection Bias in IV: Quantitative Analysis

Selection bias with truncation

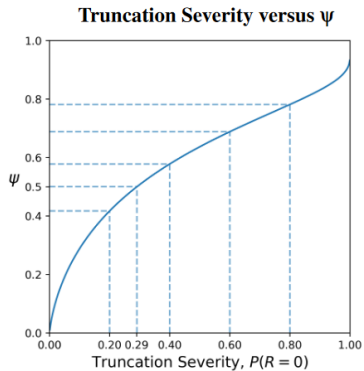
Proposition 2

In a linear and homogeneous model with normal errors represented by Figure 1a and truncation on S , $R = \mathbb{1}(S \geq s_0)$, the standard instrumental variables estimator converges in probability to

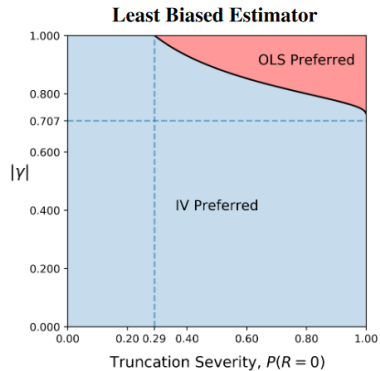
$$\beta_{IV|Tr} = \beta - \delta_1 \delta_2 \frac{\Psi \gamma^2}{1 - \Psi \gamma^2},$$

$$\text{where } \Psi = \frac{\phi(s_0)}{1 - \Phi(s_0)} \left(\frac{\phi(s_0)}{1 - \Phi(s_0)} - s_0 \right)$$

Selection bias with truncation



(a)



(b)

Selection bias with truncation

- ▶ Ψ increases with severity of truncation
 $Pr(R = 0) = \Phi(s_0)$.
- ▶ Magnitude of bias due to adjustment \geq bias due to truncation.
- ▶ Bias due to truncation approaches bias due to adjustment in the limiting case.

Selection Bias in IV: Quantitative Analysis

Comparison between adjustment and truncation

Corollary 1

In a linear and homogeneous model with normal errors represented by Figure 1a, the magnitude of IV-adjustment bias is weakly larger than that of IV-truncation bias:

$$|\beta_{IV|Adj} - \beta| \geq |\beta_{IV|Tr} - \beta|.$$

Selection Bias in IV: Quantitative Analysis

Point truncation

Proposition 2

In a linear and homogeneous model with normal errors, selection bias in the standard instrumental variables estimator due to covariate adjustment is the limiting case of selection bias due to point truncation,

$$\lim_{s_0 \rightarrow \infty} \beta_{IV|Tr} = \beta_{IV|Adj}.$$

IV vs OLS estimator

$$|\beta_{OLS|Tr} - \beta| - |\beta_{IV|Tr} - \beta| = \delta_1 \delta_2 \frac{1 - 2\Psi\gamma^2}{1 - \Psi\gamma^2}$$

- ▶ When fewer than 29.1 % of observations are truncated, IV preferred over OLS.
- ▶ For $|\gamma| \leq 0.707$, IV preferred over OLS.

Selection Bias in IV: Quantitative Analysis

Bounds for true estimate

Corollary 2

In a linear and homogeneous model with normal errors represented by Figure 1a, the OLS estimator and the instrumental variables estimator with selection bound the causal effect of T on Y , β ,

$$\beta_{IV|Tr} \leq \beta \leq \beta_{OLS|Tr}, \text{ when } \delta_1\delta_2 > 0$$

$$\beta_{IV|Tr} \geq \beta \geq \beta_{OLS|Tr}, \text{ when } \delta_1\delta_2 < 0$$

Selection bias with truncation

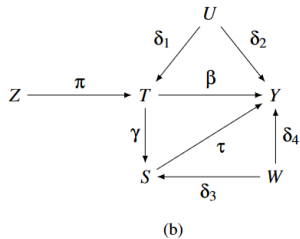
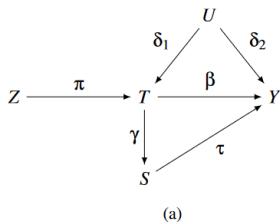


Figure: Figure 3

Selection as a Function of a Mediator

Proposition 4

In a linear and homogeneous model with normal errors represented by Figure 3a. The standard instrumental variables estimator with selection on S , converges in probability to

$$\beta_{IV|S} = \beta - \delta_1\delta_2 \frac{\Psi\gamma^2}{1 - \Psi\gamma^2} + \gamma\tau \frac{1 - \Psi}{1 - \Psi\gamma^2},$$

and the OLS estimator with selection on S converges in probability to

$$\beta_{OLS|S} = \beta + \delta_1\delta_2 + \gamma\tau \frac{1 - \Psi}{1 - \Psi\gamma^2}.$$

Selection as a Function of a Mediator

- ▶ With adjustment, the second term is zero.
- ▶ Indirect causal path $T \rightarrow S \rightarrow Y$ is blocked with adjustment.
- ▶ When truncation is small, first part close to zero, second part large.

Confounder with a mediator

- ▶ The case of figure 3b.
- ▶ With adjustment, a new path is opened
 $Z \rightarrow T \rightarrow S \leftarrow W \rightarrow Y$.
- ▶ The term $-\gamma\delta_3\delta_4\frac{\Psi}{1-\Psi\gamma^2}$.

Conclusion

- ▶ Conditioning on the wrong variable can induce selection bias.
- ▶ Bias depends on:
 - ▶ Strength of each biasing path.
 - ▶ Effect of treatment on the selection variable, $|\gamma|$.
 - ▶ Truncation severity.
- ▶ Sign of the bias may be positive or negative depending on the model and parameters.
- ▶ True causal parameter lies between IV and OLS estimator in some cases.

Question

In which cases is the true causal effect not bounded by IV and OLS estimator and why?