Direct and Indirect Effects

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Introduction

- **Total Effects:** The probability $Y = y$ when X is set to x by external intervention.

$$P(Y_x = y) = P(y|do(x))$$

- **Direct Effects:** The direct effect can be measured by holding all intermediate variables constant.

$$X \to Y$$

- **Indirect Effects:** In a linear system, indirect effect = total effect − direct effect. In nonlinear system, it cannot be measured by holding some variables constant.
- This paper presents a new way of defining the effect transmitted through a restricted set of paths without controlling variables on the remaining paths.
- This permits the assessment of a more natural type of direct and indirect effects, one that is applicable in both linear and nonlinear models and that has broader policy-related interpretations.

Example: Drug, Aspirin and Headache

$$\begin{array}{ccc} \text{Headache} & \longrightarrow & \text{Aspirin } (Z) \\ \uparrow & & \downarrow \\ \text{Drug } (X) & \longrightarrow & \text{Disease } (Y) \end{array}$$

- **Total effect:**
  How beneficial the drug is to the population as a whole.

$$P(Y_x = y) - P(Y_{x^*} = y)$$

- **Direct effect:**
  Whether aspirin should be encouraged or discouraged during the treatment.

$$P(Y_{xz} = y) - P(Y_{x^*z} = y)$$

Descriptive Versus Prescriptive Interpretation

$$\begin{array}{ccc} \text{Headache} & \longrightarrow & \text{Aspirin } (Z) \\ \uparrow & & \downarrow \\ \text{Drug } (X) & \longrightarrow & \text{Disease } (Y) \end{array}$$

- **Prescriptive:** (controlled effect)
  - Whether an untreated patient would improve if treated while holding the aspirin intake fixed at a predetermined level $Z = z$.
  - The actual consumption of aspirin under uncontrolled conditions need not concern.
  - $X = x^*$ before taking.
  - $X = x$ after taking.

$$P(Y_{xz} = y) - P(Y_{x^*z} = y)$$

- **Descriptive:** (natural effect)
  - Whether the untreated patient would improve if treated, but hold aspirin intake at the level the patient currently consumes under no treatment condition.
  - The descriptive formulation requires knowledge of the individual natural behavior.
  - Requires testing the same group of patients twice, which is rarely feasible.

$$P(Y_x = y|z) - P(Y_{x^*} = y|z)$$

Policy Implications of The Descriptive Interpretation

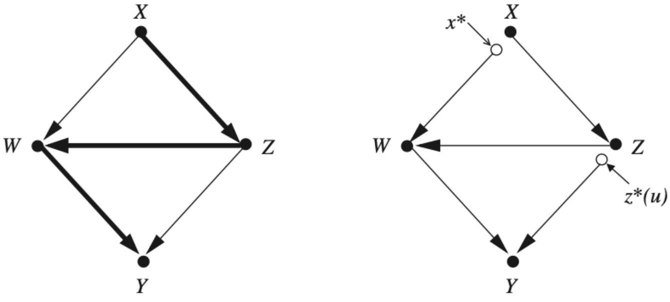
- Policy makers are more interested in **average natural direct effect**.
- In the real world, it is impossible to control all patients' aspirin usage to the same level since aspirin usage is different from individual to individual.
- The direct effect of descriptive interpretation carries operational implications and better fits the practical situation.

Descriptive Interpretation of Indirect Effects

- The descriptive conception of direct effects can be transported to the formulation of indirect effects.
- For example, to assess the natural indirect effect of a specific patient, we withhold treatment and ask whether that patient would recover if given as much aspirin as he would have taken if he had been under treatment.
- In contrast, the prescriptive formulation is not transportable since there is no way of preventing the direct effect from operating by holding certain variables constant.

General Path-Specific Effects

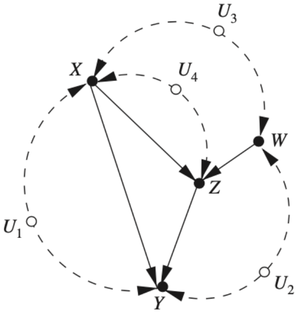
- Path-specific Effects can be understood by path-deactivation process.
- A selected set of paths are forced to remain inactive during the transition from $X = x^*$ to $X = x$
- (a) If we evaluate $X \rightarrow Z \rightarrow W \rightarrow Y$, we cannot hold Z or W constant.
- (b) We can isolate the direct effect by replacing x with x^* in the equation for W and replace z with $z^*(u) = Z_{x^*}(u)$ for Y.



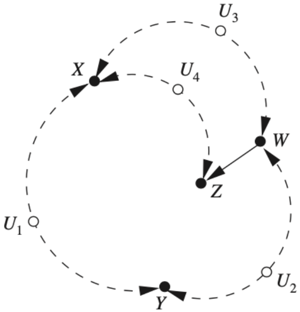
(a) (b)

Notation

- Let X be the control variable (whose effect we seek to assess)
- Let Y be the response variable.
- Let Z stand for the set of all intermediate variables between X and Y.
- Use $Y_x(u)$ to denote the value Y in unit $U = u$ under the control of $do(X = x)$.



(a)



(b)

Controlled Direct Effects (CDE) (Review)

- Given a causal model M with causal graph G, the Controlled Direct Effect (CDE) of $X = x$ on Y in unit $U = u$ and setting $Z = z$ is given by

$$CDE_z(x, x^*; Y, u) = Y_{xz}(u) - Y_{x^*z}(u)$$

- Given a probabilistic causal model $\langle M, P(u)\rangle$, the controlled direct effect of event $X = x$ on Y is defined as:

$$CDE_z(x, x^*; Y) = E(Y_{xz} - Y_{x^*z})$$

  where the expectation is taken over u.

Natural Direct Effects: Formulation

- Given a causal model M with causal graph G, the Natural Direct Effect (NDE) of $X = x$ on Y in unit $U = u$ and setting $Z = z$ is given by

$$NDE_z(x, x^*; Y, u) = Y_{xZ_{x^*}(u)}(u) - Y_{x^*}(u)$$

  $Z_{x^*}(u)$ means the z when $X = x^*$.

- Given a probabilistic causal model $\langle M, P(u) \rangle$, the average direct effect of event $X = x$ on Y is defined as:

$$NDE_z(x, x^*; Y) = E(Y_{xZ_{x^*}}) - E(Y_{x^*})$$

  where the expectation is taken over u.

Natural Direct Effects: Experimental Identification

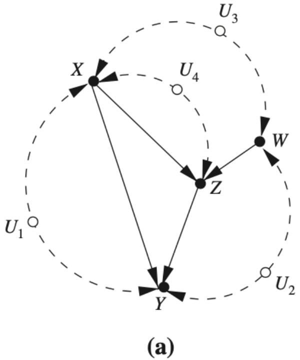
- It is difficult to evaluate the average natural direct effect from empirical data.
- Formally, this means $NDE_z(x, x^*; Y) = E(Y_{xZ_{x^*}}) - E(Y_{x^*})$ is not reducible to expressions of the form $P(Y_x = y)$ or $P(Y_{xz} = y)$
- $P(Y_x = y)$ governs the causal effect of X on Y (obtained by randomizing X).
- $P(Y_{xz} = y)$ governs the causal effect of X and Z on Y (obtained by randomizing both X and Z).
- Following pages shows the reduction is feasible.

Natural Direct Effects: Experimental Identification

- If there exists a set W of covariates, nondescendants of X or Z, such that

$$Y_{xz} \perp\!\!\!\perp Z_{x^*} | W$$

Y_{xz} is conditionally independent of Z_{x^*}, given W



Natural Direct Effects: Experimental Identification

- If there exists a set W of covariates, nondescendants of X or Z, such that

$$Y_{xz} \perp\!\!\!\perp Z_{x^*} | W$$

  Y_{xz} is conditionally independent of Z_{x^*}, given W

- The average natural direct effect is experimentally identifiable by

$$NDE_z(x, x^*; Y) = E(Y_{xZ_{x^*}}) - E(Y_{x^*})$$

$$E(Y_{xZ_{x^*}}) = \sum_w \sum_z E(Y_{xz} | Z_{x^*} = z, w) P(Z_{x^*} = z | w) P(w)$$

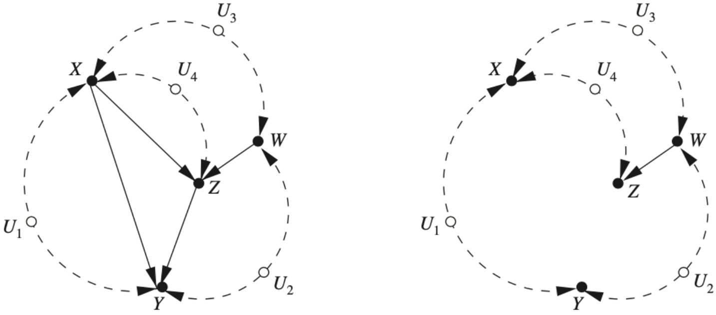
$$= \sum_w \sum_z E(Y_{xz} | w) P(Z_{x^*} = z | w) P(w)$$

$$E(Y_{x^*}) = E(Y_{x^* Z_{x^*}}) \quad \text{law of composition}$$

$$= \sum_w \sum_z E(Y_{x^* z} | w) P(Z_{x^*} = z | w) P(w)$$

$$NDE_z(x, x^*; Y) = \sum_{w,z} [E(Y_{xz} | w) - E(Y_{x^* z} | w)] P(Z_{x^*} = z | w) P(w)$$

Natural Direct Effects: Experimental Identification



(a) (b)

- (a) A causal model with latent variables (U's) where the natural direct effect can be identified in experimental studies.
- (b) The subgraph G_{XZ} illustrating the criterion of experimental identifiability: $(Y \perp\!\!\!\perp Z|W)_{G_{XZ}}$, W d-seperates Y from Z in the graph formed by deleting all (solid) arrows emanating from X and Z.

Natural Direct Effects: Nonexperimental Identification

- The identification of the natural direct effect from nonexperimental data requires stronger conditions.

$$NDE_z(x, x^*; Y) = \sum_{w,z} [E(Y_{xz}|w) - E(Y_{x^*z}|w)] P(Z_{x^*} = z|w) P(w)$$

- From the above equation, it is sufficient to identify the conditional probability if
- $Y_{xz} \perp\!\!\!\perp Z_{x^*} | W$
- $P(Y_{xz} = y | W = w)$ is identifiable
- $P(Z_{x^*} = z | W = w)$ is identifiable

Natural Direct Effects: Markovian Models

- The condition $Y_{xz} \perp\!\!\!\perp Z_{x^*}|W$ still holds when $W = \phi$.
- Since Y_{xz} is independent of all variables in the model.
- In Markovian models we have the following three relationships:
  - $P(Y_{xz} = y) = P(y|x, z)$ since $X \cup Z$ is the set of Y's parents.
  - $P(Z_{x^*} = z) = \sum_s P(z|x^*, s)P(s)$
  - $P(Y_{x,Z_{x^*}} = y) = \sum_s \sum_z P(y|x, z)P(z|x^*, s)P(s)$

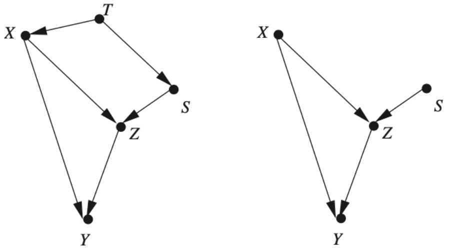
  where S stands for the parents of Z excluding X or any other set satisfying backdoor criterion.
- The average natural direct effect in Markovian models can be identifiable from nonexperimental data

$$NDE_z(x, x^*; Y) = \sum_s \sum_z [E(Y|x, z) - E(Y|x^*, z)]P(z|x^*, s)P(s)$$

- We can use simple Markovian model that the effect of X on Z is not confounded, i.e. $P(Z_{x^*} = z) = P(z|x^*)$.
- The equation can be simplified as

$$NDE_z(x, x^*; Y) = \sum \sum [E(Y|x, z) - E(Y|x^*, z)]P(z|x^*)$$

Natural Direct Effects: Markovian Models



(a) (b)

- (a) $NDE_z(x, x^*; Y) = \sum_s \sum_z [E(Y|x, z) - E(Y|x^*, z)]P(z|x^*, s)P(s)$
- (b) Simple Markovian model (X on Z is not confounded)

$$NDE_z(x, x^*; Y) = \sum_s \sum_z [E(Y|x, z) - E(Y|x^*, z)]P(z|x^*)$$

this expression can be considered as a weighted average of the controlled direct effect $E(Y|x, z) - E(Y|x^*, z)$ where the intermediate value z is chosen according to its distribution under x^*

Natural Indirect Effects: Formulation

- The Natural Indirect Effect (NIE) can be defined as

$$NIE(x, x^*; Y, u) = Y_{x^*, Z_x(u)}(u) - Y_{x^*}(u)$$

- The average indirect effect can be defined as:

$$NIE(x, x^*; Y) = E(Y_{x^*, Z_x}) - E(Y_{x^*})$$

- The Total Effect (TE) is

$$TE(x, x^*; Y) = E(Y_x) - E(Y_{x^*})$$

- We have $TE(x, x^*; Y) = NIE(x, x^*; Y) + NDE(x, x^*; Y)$

$$TE(x, x^*; Y) = NIE(x, x^*; Y) - NDE(x^*, x; Y)$$
$$TE(x, x^*; Y) = NDE(x, x^*; Y) - NIE(x^*, x; Y)$$

- The total effect on Y of the transition from x^* to x is equal to the difference between the indirect effect associated with this transition and the natural direct effect associated with the reverse transition.

Natural Indirect Effects: Identification

- If there exists a set W of covariates, nondescendants of X or Z, such that

$$Y_{x^*z} \perp\!\!\!\perp Z_x | W$$

- for all x and z, the average indirect effect is experimentally identifiable by

$$NIE(x, x^*; Y) = \sum_{w,z} E(Y_{x^*z}|w)[P(Z_x = z|w) - P(Z_{x^*} = z|w)]P(w)$$

- In non-experimental studies, the average indirect effect is identifiable when $E(Y_{x^*z}|w)$, $P(Z_x = z|w)$ and $P(Z_{x^*} = z|w)$ are identifiable.
- In a simple Markovian model, the equation can be simplified to

$$NIE(x, x^*; Y) = \sum_{z} E(Y|x^*, z)[P(z|x) - P(z|x^*)]$$

Conclusions

- This paper formulates a new definition of path-specific effects that is based on path switching.
- Instead of variable fixing, that extends the interpretation and evaluation of direct and indirect effects to nonlinear models.
- It is shown that, in nonparametric models, direct and indirect effects can be estimated consistently from both experimental and nonexperimental data, provided certain conditions hold in the causal diagram.
- Markovian models always satisfy these conditions.
- Using the new definition, the paper provides an operational interpretation of indirect effects.

Question To The Class

- Please briefly explain why the natural indirect effect is difficult to measure.