CS 295: Causal Reasoning

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More on Structural Causal Models
Definition and distributions

Primer, (chapters 1, 2) PCH 1.2

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The New Oracle: Structural Causal Models

Definition: A structural causal model (SCM) \( M \) is a 4-tuple \(<V, U, F, P(u)\>\), where

- \( V = \{V_1, \ldots, V_n\} \) are endogenous variables;
- \( U = \{U_1, \ldots, U_m\} \) are exogenous variables;
- \( F = \{f_1, \ldots, f_n\} \) are functions determining \( V \),
  \[ v_i \leftarrow f_i(pa_i, u_i), \quad Pa_i \subset V_i, U_i \subset U; \]
- \( P(u) \) is a distribution over \( U \)

Axiomatic Characterization:

(Galles-Pearl, 1998; Halpern, 1998).
2. SCM → Causal Diagram

• Every SCM $M$ induces a causal diagram

• Represented as a DAG where:
  • Each $V_i \in V$ is a node,
  • There is $W \rightarrow V_i$ if for $W \in Pa_i$,
  • There is $V_i \leftarrow \cdots \leftarrow V_j$ whenever $U_i \cap U_j \neq \emptyset$.

\[
V_i \leftarrow f_i (A,B,U) \\
V_j \leftarrow f_j (C,U)
\]
2. SCM → Causal Diagram

Recall our medical example:

- **Endogenous (observed) variables** $V$:
  - $R$ ($R=1$ for rich, $=0$ for poor)
  - $D$ ($D=1$ for taking the drug, $D=0$ o/w)
  - $A$ ($A=1$ if person ends up alive, $=0$ o/w)

- **Exogenous (unobserved) Variables** $U$:
  - $U_r$ (Wealthiness factors)
  - $U_g$ ($=1$ has the genetic factor, $=0$ o/w)

- Distribution over $U$: $P(U_r)=1/2$, $P(U_g)=1/2$
2. SCM $\rightarrow$ Causal Diagram

Another example:

- $V = \{\text{Smoking}, \text{Cancer}\}$
- $U = \{U_s, U_c, U_g\}$
- $\mathcal{F}$ (unobserved genotype)

\[
\text{Smoking} \leftarrow f_{\text{Smoking}}(U_s, U_g) \quad \text{Cancer} \leftarrow f_{\text{Cancer}}(\text{Smoking}, U_c, U_g)
\]

Remark 1. The mapping is just 1-way (i.e., from a SCM to a causal graph) since the graph itself is compatible with infinitely many SCMs with the same scope (the same functions signatures and compatible exogenous distributions).

Remark 2. This observation will be central to causal inference since, in most practical settings, researchers may know the scope of the functions, for example, but not the details about the underlying mechanisms.
SCM inspires structure on $P(V,U)$

Does the causal diagram give us any clues about the (in)dependence relations in the obs. distribution $P(V)$?

- Is $T$ independent of $W$?
- Is $W$ independent of $T$?
- Is $Z$ independent of $T$?
- Is $Z$ independent of $X$?
- Is $Y$ independent of $W$?
- Is $Y$ independent of $W$ if we know the value of $X$?

$$M = \begin{cases} 
Z & \leftarrow f_Z(u_z) \\
\mathcal{W} & \leftarrow f_W(z, x, u_w) \\
& f_X(u_x) f_Y(x, u_y) \\
T & \leftarrow f_T(w, u_t) 
\end{cases}$$

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3. SCM → Pearl’s Causal Hierarchy

<table>
<thead>
<tr>
<th>Level (Symbol)</th>
<th>Typical Activity</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Association $P(y \mid x)$</td>
</tr>
<tr>
<td></td>
<td>Seeing ML - (Un)Supervised Learning</td>
</tr>
<tr>
<td>2</td>
<td>Intervention $P(y \mid do(x), c)$</td>
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<tr>
<td>3</td>
<td>Counterfactual $P(y_x \mid x', y')$</td>
</tr>
<tr>
<td></td>
<td>Doing ML - Reinforcement Learning</td>
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<tr>
<td></td>
<td>Imagining, Retrospection</td>
</tr>
</tbody>
</table>
Outline

• Structural Causal Models
• Product form of Markov SCM
• d-seperation
• Bayesian networks
The Emergence of the First Layer

In our example,

\[ P(R = r, D = d, A = a) = \sum_{u_r, u_g} P(R = r, D = d, A = a, U_r = u_r, U_g = g) \]

For short,

\[ P(r, d, a) = \sum_{u_r, u_g} P(r, d, a, u_r, u_g) \]
The Emergence of the First Layer

In the second example,

\[ P(s, c) = \sum_{u_s, u_g, u_c} P\left(s, c, u_s, u_g, u_c\right) \]

The joint probability distribution over the observed variables (V), Smoking and Cancer, is given by

Recall, this distribution is called observational distribution. Sometimes, it’s also called passive or non-experimental distribution.
What the Diagram Encodes

• Since $G$ is a directed acyclic graph, there exists a topological order over $V$ such that every variable goes after its parents, i.e., $Pa_i < V_i$.

$$\begin{align*}
    R & < D < A \\
    S & < C
\end{align*}$$
What the Diagram Encodes

- \( M \) induces \( P(V) \):
  \[
P(v) = \sum_u P(v, u),
  \]
  - Using the chain rule and following a topological order,
    \[
P(v) = \sum_u P(u) \prod_{v \in V} P(v_i \mid v_1, \ldots, v_{i-1}, u),
  \]
- An observed variable is fully determined by its observed and unobserved parents; also \( \{pa_i, u_i\} \subseteq \{v_i, \ldots, v_{i-1}, u\} \), then
  \[
P(v_i \mid v_1, \ldots, v_{i-1}, u) = P(v_i \mid pa_i, u).
  \]
What the Diagram Encodes

- The distribution $P(V)$ decomposes as:

  $$P(v) = \sum_u P(u) \prod_{V_i \in V} P(v_i | v_1, \ldots, v_{i-1}, u)$$

  $$= \sum_u P(u) \prod_{V_i \in V} P(v_i | pa_i, u_i)$$

  $$P(r, d, a) = \sum_{u_r, u_g} P(u_r, u_g)P(r | u_r)P(d | r)P(a | r, d, u_g)$$

  $$P(s, c) = \sum_{u_s, u_g, u_c} P(u_s, u_g, u_c)P(s | u_g, u_s)P(c | s, u_g, u_c)$$
Conditional Independences

- If knowing that variable $X = x$ doesn’t change the belief in $Y = y$, then $X$ and $Y$ are said to be *probabilistically independent*. This is written as $X \perp Y$.

  $\text{• } X \perp Y \equiv P(Y = y | X = x) = P(Y = y)$

- More generally, once we know the value of a third variable $Z = z$, if knowing that $X = x$ doesn’t affect the belief of $Y = y$, $X$ and $Y$ are *conditionally independent* given $Z$, i.e., $X \perp Y | Z$.

  $\text{• } X \perp Y | Z \equiv P(X = x | Z = z, Y = y) = P(X = x | Z = z)$.

*Lack of functional dependence $\rightarrow$ probabilistic independence.*
Markovian Factorization

• Suppose no variable in \( U \) is a parent of two variables in \( V \) (observables) (i.e., \( \forall_{i,j} U_i \cap U_j = \emptyset \)), then the model is called Markovian. We have:

\[
P(v) = \sum_u P(u) \prod_{v_i \in V} P(v_i | pa_i, u_i) \\
= \sum_u \prod_{v_i \in V} P(v_i | pa_i, u_i) P(u_i) \\
= \sum_u \prod_{v_i \in V} P(v_i | pa_i, u_i) P(u_i | pa_i) \\
= \prod_{v_i \in V} \sum_{u_i} P(v_i, u_i | pa_i) = \prod_{v_i \in V} P(v_i | pa_i)
\]

Local Markovian Condition

\( (V_i \perp N_{di} \mid Pa_i \mid Pa_i) \)
Outline

• Structural Causal Models
• Product form of Markov SCM
• d-separation: reading independencies from the DAG
• Bayesian networks
Causal Chains

• Are $X$ and $Y$ independent given $Z$?

• Yes,

• e.g., knowing $Z=1$ (high SAT scores), the probability of being admitted ($Y=1$) does not change if we know the student has good study habits ($X=1$) or not ($X=0$).

$$P(x, y | z) = \frac{P(x, z, y)}{P(z)} = \frac{P(x) P(z | x) P(y | z)}{P(z)}$$

Bayes Factorization

Graphically, observing $Z$ “blocks” the influence from $X$ to $Y$.
Common Cause

- Are \( X \) and \( Y \) independent?
- No,
  - e.g., seeing someone coming in with an umbrella in hand \((Y=1)\) rises the probability of rain \((Z=1)\), which increases the likelihood of bad traffic \((X=1)\).

Graphically, information “flows” from \( Y \) going through the common cause \( Z \) and down to \( X \).

\[
M: \begin{align*}
Z &\leftarrow U_z \\
X &\leftarrow Z \oplus \neg U_x \\
Y &\leftarrow Z \lor U_y \\
P(U_x, U_z, U_y) &
\end{align*}
\]

\[\exists_{x,y} P(X = x, Y = y) \neq P(X = x)P(Y = y) \text{ try it out!}\]
Are $X$ and $Y$ independent given $Z$?

Yes,
e.g., if we know it is raining ($Z=1$), observing people with umbrellas ($Y=1$) tells us nothing about the traffic ($X$).

Bayes Factorization

$$P(x, y \mid z) = \frac{P(x, z, y)}{P(z)} = \frac{P(z) P(x \mid z) P(y \mid z)}{P(z)} = P(x \mid z) P(y \mid z)$$

Graphically, observing $Z$ “blocks” the influence from $X$ to $Y$.

$M$: $Z \leftarrow U_z$
$X \leftarrow Z \oplus \neg U_x$
$Y \leftarrow Z \lor U_y$
$P(U_x, U_z, U_y)$
Common Effect

- Are $X$ and $Y$ independent? Yes!

- e.g., having a cold $X=1$ is independent of being on holiday $Y=1$.

\[
P(x, y) = \sum P(x)P(y)P(z \mid x, y) = P(x)P(y) \sum P(z \mid x, y) = P(x)P(y)
\]

Graphically, influence from $X$ reaches $Z$ but does not “go up” to $Y$.

\[
M: X \leftarrow U_x \\
Y \leftarrow U_y \\
Z \leftarrow \neg Y \land (\neg X \oplus U_z)
\]

\[
P(U_x, U_z, U_y)
\]
Common Effect

Are $X$ and $Y$ independent given $Z$?

No!

e.g., if we observe that a student didn’t go to class ($Z=0$) and today is not a holiday ($Y=0$), it is more likely that she may have a cold ($X=1$).

$\exists_{x,y,z} P(X = x, Y = y \mid Z = z) \neq P(X = x \mid Z = z) P(Y = y \mid Z = z)$

try it out!

Graphically, influence from $X$ reaching $Z$ (when $Z$ is observed) bumps “back up” to $Y$.

This behavior is opposite to the previous cases.
Common Effect

- Are $X$ and $Y$ independent given $W$? No!, again!

- e.g., observing that the student didn’t sign the assistance sheet ($W=0$) increases the likelihood of the student being absent ($Z=0$), that as we said, make $X$ and $Y$ dependent.

Graphically, influence from $X$ reaching $W$ (when $W$ is observed) “bumps back up” to $Z$, and then $Y$.

Watch out for the descendants of the colliders!
Summary

Active Triplets

Normal

\[ X \rightarrow Z \rightarrow Y \]
\[ X \leftarrow Z \leftarrow Y \]
\[ X \leftarrow Z \rightarrow Y \]
\[ (X \not\perp Y) \]

Abnormal

\[ X \rightarrow Z \rightarrow Y \]
\[ (X \not\perp Y | Z) \]

\[ X \leftarrow Z \rightarrow Y \]
\[ (X \not\perp Y | Z) \]

\[ X \rightarrow Z \rightarrow Y \]
\[ (X \not\perp Y | Z) \]

Inactive Triplets

\[ X \rightarrow Z \rightarrow Y \]
\[ (X \perp Y | Z) \]

\[ X \leftarrow Z \rightarrow Y \]
\[ (X \perp Y | Z) \]

\[ X \leftarrow Z \rightarrow Y \]
\[ (X \perp Y) \]

What about larger graphical structures?
Graph Separation (d-Separation)

- Consider the question of whether $X$ and $Y$ are independent given $Z$.

1. Look at every path from $X$ to $Y$ in the graph.

2. A path is active if every triplet in it is active (given $Z$).

3. If any path is active, $X$ and $Y$ are not independent.
Graph Separation (d-Separation)

$\text{CI}_1$: $(\text{Wet} \perp \text{Sprinkler})$

$\text{CI}_2$: $(\text{Wet} \perp \text{Season} \mid \text{Sprinkler})$

$\text{CI}_3$: $(\text{Rain} \perp \text{Slippery} \mid \text{Wet})$

$\text{CI}_4$: $(\text{Season} \perp \text{Wet} \mid \text{Sprinkler, Rain})$

$\text{CI}_5$: $(\text{Sprinkler} \perp \text{Rain} \mid \text{Season, Wet})$
Graph Separation (d-Separation)

✓ CI₁: \((\text{Wet} \perp \text{Sprinkler})\)

✓ CI₂: \((\text{Wet} \perp \text{Season} \mid \text{Sprinkler})\)

✓ CI₃: \((\text{Rain} \perp \text{Slippery} \mid \text{Wet})\)

✓ CI₄: \((\text{Season} \perp \text{Wet} \mid \text{Sprinkler}, \text{Rain})\)

✓ CI₅: \((\text{Sprinkler} \perp \text{Rain} \mid \text{Season}, \text{Wet})\)
d-Separation (food for thought)

- Is A independent of D?
- Is A independent of C?
- Is A independent of C given D?
- Is D independent of C given B?

We want to be able to answer all these questions just from the DAG.
Outline

• Structural Causal Models
• Product form of Markov SCM
• d-seperation
• Bayesian networks
• More of d-seperation
DECOMPOSITION BY BAYESIAN NETWORKS

Given a distribution $P$, on $n$ discrete variables, $X_1, X_2, \ldots, X_n$. Decompose $P$ by the chain rule:

$$P(x_1, \ldots, x_n) = \prod_j P(x_j|x_1, \ldots, x_{j-1}). \quad (1.30)$$

Suppose $X_j$ is independent of all other predecessors, once we know the value of a select group of predecessors called $PA_j$. Simplification:

$$P(x_j|x_1, \ldots, x_{j-1}) = P(x_j|pa_j) \quad (1.31)$$

$PA_j$: Markovian parents of $X_j$, relative to a given ordering.
A **Bayesian network** is:

- An **directed acyclic graph (DAG)**, where
- Each node represents a random variable
- And is associated with the conditional probability of the node given its parents.
Bayesian Networks: Representation

\[
P(S, C, B, X, D) = P(S) P(C|S) P(B|S) P(X|C,S) P(D|C,B)
\]

Conditional Independencies → Efficient Representation
Definition 1.2.1 (Markovian Parents)

Let \( V = \{X_1, \ldots, X_n\} \) be an ordered set of variables, and let \( P(v) \) be the joint probability distribution on these variables. A set of variables \( PA_j \) is said to be Markovian parents of \( X_j \) if \( PA_j \) is a minimal set of predecessors of \( X_j \) that renders \( X_j \) independent of all its other predecessors. In other words, \( PA_j \) is any subset of \( \{X_1, \ldots, X_{j-1}\} \) satisfying

\[
P(x_j|pa_j) = P(x_j|x_1, \ldots, x_{j-1}) \quad (1.32)
\]

and such that no proper subset of \( PA_j \) satisfies (1.32).

**Interpretation:**

Knowing the values of other preceding variables is redundant once we know the values \( pa_j \) of the parent set \( PA_j \).
CONSTRUCTING A BAYESIAN NETWORK

Given: $P$, and an ordering of the variables.

At the $j$th stage, select any minimal set of $X_j$'s predecessors that screens off $X_j$ from its other predecessors.

Call this set $PA_j$, and draw an arrow from each member in $PA_j$ to $X_j$.

The result is a directed acyclic graph, called a Bayesian network, in which an arrow from $X_i$ to $X_j$ assigns $X_i$ as a Markovian parent of $X_j$, consistent with Definition 1.2.1.

The resulting network is unique given the ordering of the variables, whenever the distribution $P(v)$ is strictly positive.
Bayesian Networks: Representation

\[ P(S, C, B, X, D) \]

Is X independent of B given S?
MARKOV COMPATIBILITY

Definition 1.2.2 (Markov Compatibility)
If a probability function $P$ admits the factorization of (1.33) relative to DAG $G$, we say that $G$ represents $P$, that $G$ and $P$ are compatible, or that $P$ is Markov relative to $G$.

Compatibility implies that $G$ can “explain” the generation of the data represented by $P$. 
THE $d$-SEPARATION CRITERION

Definition 1.2.3 ($d$-Separation)

A path $p$ is said to be $d$-separated (or blocked) by a set of nodes $Z$ if and only if

1. $p$ contains a chain $i \rightarrow m \rightarrow j$ or a fork $i \leftarrow m \rightarrow j$
   such that the middle node $m$ is in $Z$, or

2. $p$ contains an inverted fork (or collider) $i \rightarrow m \leftarrow j$
   such that the middle node $m$ is not in $Z$ and such that no descendant of $m$ is in $Z$.

A set $Z$ is said to $d$-separate $X$ from $Y$ if and only if $Z$ blocks every path from a node in $X$ to a node in $Y$.  

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Theorem 1.2.4
(Probabilistic Implications of $d$-Separation)
If sets $X$ and $Y$ are $d$-separated by $Z$ in a DAG $G$, then $X$ is independent of $Y$ conditional on $Z$ in every distribution compatible with $G$. Conversely, if $X$ and $Y$ are not $d$-separated by $Z$ in a DAG $G$, then $X$ and $Y$ are dependent conditional on $Z$ in at least one distribution compatible with $G$.

Theorem 1.2.5
For any three disjoint subsets of nodes $(X,Y,Z)$ in a DAG $G$ and for all probability functions $P$, we have:

(i) $(X \perp Y | Z)_G \implies (X \perp Y | Z)_P$ whenever $G$ and $P$ are compatible, and

(ii) if $(X \perp Y | Z)_P$ holds in all distributions compatible with $G$, it follows that $(X \perp Y | Z)_G$. 

G is an Independency map (IMAP) of any compatible $P$ relative to $d$-separation.
Theorem 1.27
(Parental Markov Condition)
A necessary and sufficient condition for a probability distribution $P$ to be Markov relative a DAG $G$ is that every variable be independent of all its nondescendants (in $G$), conditional on its parents.

Theorem 1.28
(Observational Equivalence)
Two DAGs are observationally equivalent if and only if they have the same skeletons and the same sets of $v$-structures, that is, two converging arrows whose tails are not connected by an arrow (Verma and Pearl 1990).

Will discuss later
Outline

• Structural Causal Models
• Product form of Markov SCM
• d-separation
• Bayesian networks
• More of d-separation
Triplets - Summary

Active Triplets

\[ X \rightarrow Z \rightarrow Y \]
\[ X \rightarrow Z \rightarrow Y \]
Triplets - Summary

Active Triplets

\[
\begin{align*}
X & \rightarrow Z & \rightarrow Y \\
X & \leftarrow Z & \rightarrow Y \\
X & \rightarrow Z & \leftarrow Y \\
\end{align*}
\]
Triplets - Summary

Active Triplets

\[ X \rightarrow Z \rightarrow Y \]
\[ X \leftarrow Z \rightarrow Y \]
\[ X \rightarrow Z \leftarrow Y \]
\[ X \rightarrow Z \leftarrow Y \]
\[ X \leftarrow \downarrow \rightarrow \]
\[ W \]
**Triplets - Summary**

**Active Triplets**

\[
(X \gets Y | Z)
\]

\[
(X \leftrightarrow Y | Z)
\]

\[
(X \rightarrow Z \leftarrow Y)
\]

\[
(X \rightarrow Z \rightarrow Y)
\]
Triples - Summary

Active Triplets

\[ X \rightarrow Z \rightarrow Y \]
\[ X \leftarrow Z \rightarrow Y \]
\[ X \rightarrow Z \leftarrow Y \]
\[ X \rightarrow Z \leftarrow Y \downarrow \]
\[ (X \perp Y \mid Z) \]

Inactive Triplets

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Triplets - Summary

Active Triplets

\[ X \rightarrow Z \rightarrow Y \]
\[ X \leftrightarrow Z \rightarrow Y \]
\[ X \rightarrow Z \leftrightarrow Y \]
\[ X \rightarrow Z \leftarrow Y \]
\[ (X \parallel Y \mid Z) \]

Inactive Triplets

\[ X \rightarrow Z \rightarrow Y \]
Triplets - Summary

Active Triplets

\[ X \rightarrow Z \rightarrow Y \]
\[ X \leftrightsquigarrow Z \rightarrow Y \]
\[ X \rightarrow Z \leftrightsquigarrow Y \]
\[ X \rightarrow Z \rightarrow Y \]
\[ W \]

\( (X \parallel Y \mid Z) \)

Inactive Triplets

\[ X \rightarrow Z \rightarrow Y \]
\[ X \leftrightsquigarrow Z \rightarrow Y \]
\[ X \rightarrow Z \leftrightsquigarrow Y \]
Triplets - Summary

Active Triplets

- $X \rightarrow Z \rightarrow Y$
- $X \leftarrow Z \rightarrow Y$
- $X \leftarrow Z \leftarrow Y$
- $X \rightarrow Z \leftarrow Y$
- $W$

Inactive Triplets

- $X \rightarrow Z \rightarrow Y$
- $X \leftarrow Z \rightarrow Y$
- $X \rightarrow Z \leftarrow Y$

$(X \not\perp Y \mid Z)$
Triples - Summary

Active Triplets

$X \rightarrow Z \rightarrow Y$

$X \leftarrow Z \rightarrow Y$

$X \rightarrow Z \leftarrow Y$

$X \rightarrow Z \leftarrow Y$

$(X \parallel Y | Z)$

Inactive Triplets

$X \rightarrow Z \rightarrow Y$

$X \leftarrow Z \rightarrow Y$

$X \rightarrow Z \leftarrow Y$

$(X \parallel Y | Z)$
Triplets - Summary

Active Triplets

\[ X \rightarrow Z \rightarrow Y \]
\[ X \leftarrow Z \rightarrow Y \]
\[ X \rightarrow Z \leftarrow Y \]
\[ X \rightarrow Z \leftarrow Y \]
\[ (X \perp Y | Z) \]

Inactive Triplets

\[ X \rightarrow Z \rightarrow Y \]
\[ X \leftarrow Z \rightarrow Y \]
\[ X \rightarrow Z \leftarrow Y \]
\[ (X \perp Y | Z) \]

What about larger graphical structures?
Graph Separation (d-Separation)

• Consider the question of whether $X$ and $Y$ are independent given $Z$.

1. Look at every path from $X$ to $Y$ in the graph.

2. A path is active if every triplet in it is active (given $Z$).

3. If any path is active $X$ and $Y$ are not $d$-separated.
Graph Separation (d-Separation)

\((\text{Wet} \perp \perp \text{Sprinkler})?)\?
Graph Separation (d-Separation)

(Wet $\perp$ Sprinkler)?

Path 1: $Sp \rightarrow We$
Graph Separation (d-Separation)

(Wet ⊥ Sprinkler)?

Path 1: $Sp \rightarrow We$
Graph Separation (d-Separation)

(Wet \perp Sprinkler)^

Path 1: $Sp \rightarrow We$ always active
Graph Separation (d-Separation)

(Wet $\perp$ Sprinkler)?

Path 1: $Sp \rightarrow We$ always active

Path 2: $Sp \leftarrow Se \rightarrow Ra \rightarrow We$
Graph Separation (d-Separation)

(Wet \parallel\parallel Sprinkler)?

Path 1: \(Sp \rightarrow We\) always active

Path 2: \(Sp \leftarrow Se \rightarrow Ra \rightarrow We\)
Graph Separation (d-Separation)

\((Wet \perp \!\!\!\perp Sprinkler)\)?

Path 1: \(Sp \rightarrow We\) always active

Path 2: \(Sp \leftarrow Se \rightarrow Ra \rightarrow We\)
Graph Separation (d-Separation)

\((Wet \perp\!\!\!\!\!\!\!\perp Sprinkler)\)?

Path 1: \(Sp \rightarrow We\) \hspace{1cm} always active

Path 2: \(Sp \leftarrow Se \rightarrow Ra \rightarrow We\)

There exists a path (actually two) that is active, hence \textit{Sprinkler} and \textit{Wet} are not d-separated.
Graph Separation (d-Separation)

(Wet ⊥ Season | Sprinkler)?
Graph Separation (d-Separation)

(Wet $\perp$ Season | Sprinkler)?

Path 1: $Se \rightarrow Sp \rightarrow We$
Graph Separation (d-Separation)

$(Wet \perp\!\!\!\perp Season \mid Sprinkler)$?

Path 1: $Se \rightarrow Sp \rightarrow We$
Graph Separation (d-Separation)

$(\text{Wet} \perp \text{Season} \mid \text{Sprinkler})$?

Path 1: $\text{Se} \rightarrow \text{Sp} \rightarrow \text{We}$
Graph Separation (d-Separation)

\((Wet \perp \!\!\!\!\!\!\!\!\perp \text{Season \mid Sprinkler})?\)

Path 1: \(Se \rightarrow Sp \rightarrow We\)

Path 2: \(Se \rightarrow Ra \rightarrow We\)
Graph Separation (d-Separation)

(Wet \perp Season \mid Sprinkler)\?

Path 1: $Se \rightarrow Sp \rightarrow We$

Path 2: $Se \rightarrow Ra \rightarrow We$
Graph Separation (d-Separation)

\((\text{Wet} \perp \perp \text{Season} \mid \text{Sprinkler})?\)

Path 1: \(\text{Se} \rightarrow \text{Sp} \rightarrow \text{We}\)

Path 2: \(\text{Se} \rightarrow \text{Ra} \rightarrow \text{We}\)

There exists a path that is active, hence \textit{Wet} and \textit{Season} are not d-separated given \textit{Sprinkler}.
Graph Separation (d-Separation)

\[(\text{Rain} \independent \text{Slippery} \mid \text{Wet})?\]
Graph Separation (d-Separation)

(Rain \perp Slippery \mid Wet)?

Path 1: \textbf{Ra} \rightarrow \textbf{We} \rightarrow \textbf{Sl}
Graph Separation (d-Separation)

(Rain \perp Slippery \mid Wet)?

Path 1: \[ Ra \rightarrow We \rightarrow Sl \]
Graph Separation (d-Separation)

(Rain \perp Slippery \mid Wet)?

Path 1: \textcolor{red}{Ra} \rightarrow \textcolor{red}{We} \rightarrow \textcolor{red}{Sl}
Graph Separation (d-Separation)

(Rain \perp\!\!\!\!\!\!\!\!\!\perp Slippery \mid Wet)?

Path 1: \[ Ra \rightarrow We \rightarrow Sl \]

Path 2: \[ Ra \leftarrow Se \rightarrow Sp \rightarrow We \rightarrow Sl \]
Graph Separation (d-Separation)

\[(\text{Rain} \perp \text{Slippery} \mid \text{Wet})?\]

Path 1: \( Ra \rightarrow We \rightarrow Sl \)

Path 2: \( Ra \leftarrow Se \rightarrow Sp \rightarrow We \rightarrow Sl \)
Graph Separation (d-Separation)

(Rain \perp Slippery \mid Wet)?

Path 1: \(Ra \rightarrow We \rightarrow Sl\)

Path 2: \(Ra \leftarrow Se \rightarrow Sp \rightarrow We \rightarrow Sl\)
Graph Separation (d-Separation)

(Rain $\perp\!\!\!\!\perp$ Slippery | Wet)?

Path 1: $Ra \rightarrow We \rightarrow Sl$

Path 2: $Ra \leftarrow Se \rightarrow Sp \rightarrow We \rightarrow Sl$
Graph Separation (d-Separation)

\((\text{Rain} \perp \text{Slippery} \mid \text{Wet})\)?

Path 1: \(\text{Ra} \longrightarrow \text{We} \rightarrow \text{Sl}\)

Path 2: \(\text{Ra} \leftarrow \text{Se} \longrightarrow \text{Sp} \longrightarrow \text{We} \rightarrow \text{Sl}\)
Graph Separation (d-Separation)

(Rain \perp Slippery \mid Wet)?

Path 1: \( Ra \rightarrow We \rightarrow Sl \)

Path 2: \( Ra \leftarrow Se \rightarrow Sp \rightarrow We \rightarrow Sl \)

There exists \textbf{no} path that is active between \textit{Rain} and \textit{Slippery} given \textit{Wet}, hence they are d-separated.
Graph Separation (d-Separation)

$(Sprinkler \perp\!
\!
\perp Rain \mid Season, Wet)$?
Graph Separation (d-Separation)

\((Sprinkler \perp\!\!\!\perp Rain \mid Season, Wet)?)\]

Path 1:  \(Sp \leftarrow Se \rightarrow Ra\)
Graph Separation (d-Separation)

(Sprinkler \perp\!
\!
\perp Rain \mid Season, Wet)?

Path 1: \( Sp \leftarrow Se \rightarrow Ra \\)
Graph Separation (d-Separation)

(Sprinkler \perp\!\!\!\!\!\!\perp Rain \mid Season, Wet)?

Path 1: $Sp \leftarrow Se \rightarrow Ra$

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Graph Separation (d-Separation)

\((Sprinkler \perp\!\!\!\perp Rain \mid Season, Wet)\)?

Path 1: \( Sp \leftarrow Se \rightarrow Ra \)

Path 2: \( Sp \rightarrow We \leftarrow Ra \)
Graph Separation (d-Separation)

\[(Sprinkler \perp \!
\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\![\text{Ra}]

Path 1: \(Sp \leftarrow Se \rightarrow \text{Ra}\)

Path 2: \(Sp \rightarrow We \leftarrow \text{Ra}\)
Graph Separation (d-Separation)

(Sprinkler \perp\!\!\!\!\!\perp Rain \mid Season, Wet)?

Path 1: $Sp \xleftarrow{Ra} Se \rightarrow Ra$

Path 2: $Sp \rightarrow We \xleftarrow{Ra}$ becomes active given $We$
Graph Separation (d-Separation)

\((Sprinkler \perp \!\!\!\perp Rain \mid Season, Wet)\)?

Path 1: \(Sp \leftarrow Se \rightarrow Ra\)

Path 2: \(Sp \rightarrow We \leftarrow Ra\) becomes active given \(We\)

There exists a path that is active between \(Sprinkler\) and \(Rain\) given \(Season\) and \(Wet\), hence they are not \(d\)-separated.
Graph Separation (d-Separation)

Is there a set $A$ such that the separation statement
$(W \perp Z \mid A)$ holds?
Graph Separation (d-Separation)

Is there a set \( A \) such that the separation statement 
\((W \perp Z \mid A)\) holds?

Path 1: \( W \leftarrow T \rightarrow Z \)
Graph Separation (d-Separation)

Is there a set $A$ such that the separation statement $(W \perp Z \mid A)$ holds?

Path 1: $W \leftarrow T \rightarrow Z$

Path 2: $W \rightarrow X \rightarrow Z$
Graph Separation (d-Separation)

Is there a set $A$ such that the separation statement $(W \perp Z \mid A)$ holds?

Path 1: $W \leftarrow T \rightarrow Z$

Path 2: $W \rightarrow X \rightarrow Z$

Path 3: $W \rightarrow X \leftrightarrow Y \leftrightarrow Z$
Is there a set $A$ such that the separation statement $(W \perp Z \mid A)$ holds?

Path 1: $W \leftarrow T \rightarrow Z$

Path 2: $W \rightarrow X \rightarrow Z$

Path 3: $W \rightarrow X \leftrightarrow Y \leftarrow Z$
Graph Separation (d-Separation)

Is there a set $A$ such that the separation statement $(W \perp Z \mid A)$ holds?

Path 1: $W \leftarrow T \rightarrow Z$

Path 2: $W \rightarrow X \rightarrow Z$

Path 3: $W \rightarrow X \leftarrow Y \leftarrow Z$

Path 4: $W \rightarrow X \leftarrow U \rightarrow Y \leftarrow Z$

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Graph Separation (d-Separation)

Is there a set $A$ such that the separation statement $(W \perp Z \mid A)$ holds?

Path 1: $W \leftrightarrow T \rightarrow Z$

Path 2: $W \rightarrow X \rightarrow Z$

Path 3: $W \rightarrow X \leftrightarrow Y \leftrightarrow Z = W \rightarrow X \leftarrow U \rightarrow Y \leftarrow Z$
Graph Separation (d-Separation)

Is there a set \( A \) such that the separation statement \((W \perp Z \mid A)\) holds?

Path 1: \( W \leftarrow T \rightarrow Z \)
Path 2: \( W \rightarrow X \rightarrow Z \)
Path 3: \( W \rightarrow X \leftrightarrow Y \leftarrow Z \)

Path 1 and 2 need to be blocked, Path 3 is naturally blocked: \( A=\{T, X\} \) suffices.

\[ W \rightarrow X \leftrightarrow U \rightarrow Y \leftarrow Z \]
Graph Separation (d-Separation)

Is there a set $A$ such that the separation statement $(R, Z \perp S \mid A)$ holds?
Graph Separation (d-Separation)

Is there a set $A$ such that the separation statement $(R, Z \perp S \mid A)$ holds?

Path 1: $R \rightarrow Y \rightarrow S$
Graph Separation (d-Separation)

Is there a set $A$ such that the separation statement $(R, Z \perp S \mid A)$ holds?

Path 1: $R \rightarrow Y \rightarrow S$

Path 2: $Z \rightarrow Y \rightarrow S$
Graph Separation (d-Separation)

Is there a set $A$ such that the separation statement $(R, Z \perp S \mid A)$ holds?

Path 1: $R \rightarrow Y \rightarrow S$

Path 2: $Z \rightarrow Y \rightarrow S$

Path 3:
$Z \leftarrow X \leftrightarrow Y \rightarrow S$
Graph Separation (d-Separation)

Is there a set $A$ such that the separation statement $(R, Z \perp S \mid A)$ holds?

Path 1: $R \rightarrow Y \rightarrow S$

Path 2: $Z \rightarrow Y \rightarrow S$

Path 3:
$Z \leftarrow X \leftrightarrow Y \rightarrow S$

Path 4: $Z \leftarrow T \rightarrow W \rightarrow X$
$\leftrightarrow Y \rightarrow S$
Graph Separation (d-Separation)

Is there a set $A$ such that the separation statement $(R, Z \perp S \mid A)$ holds?

Path 1: $R \rightarrow Y \rightarrow S$

Path 2: $Z \rightarrow Y \rightarrow S$

Path 3: $Z \leftarrow X \leftrightarrow Y \rightarrow S$

Path 4: $Z \leftarrow T \rightarrow W \rightarrow X \leftrightarrow Y \rightarrow S$
Graph Separation (d-Separation)

Is there a set $A$ such that the separation statement $(R, Z \perp S \mid A)$ holds?

Path 1: $R \rightarrow Y \rightarrow S$

Path 2: $Z \rightarrow Y \rightarrow S$

Path 3:
$Z \leftarrow X \leftrightarrow Y \rightarrow S$

Path 4: $Z \leftarrow T \rightarrow W \rightarrow X \leftrightarrow Y \rightarrow S$

$A=\{Y\}$ suffices.
Graph Separation (d-Separation)

Is there a set $A$ such that the separation statement $(W \perp Y \mid A)$ holds?
Graph Separation (d-Separation)

Is there a set $A$ such that the separation statement $(W \perp Y \mid A)$ holds?

Path 1: $W \rightarrow X \rightarrow Z \rightarrow Y$
Graph Separation (d-Separation)

Is there a set $A$ such that the separation statement $(W \perp Y \mid A)$ holds?

Path 1: $W \rightarrow X \rightarrow Z \rightarrow Y$

{X} or {Z} or {X, Z}
Graph Separation (d-Separation)

Is there a set $A$ such that the separation statement $(W \perp Y \mid A)$ holds?

Path 1: $W \rightarrow X \rightarrow Z \rightarrow Y$

Path 2: $W \leftarrow T \rightarrow Z \rightarrow Y$

Set: $\{X\}$ or $\{Z\}$ or $\{X, Z\}$
Graph Separation (d-Separation)

Is there a set $A$ such that the separation statement $(W \perp Y \mid A)$ holds?

Path 1: $W \rightarrow X \rightarrow Z \rightarrow Y$

Path 2: $W \leftarrow T \rightarrow Z \rightarrow Y$

$\{X\}$ or $\{Z\}$ or $\{X, Z\}$

$\{T\}$ or $\{Z\}$ or $\{T, Z\}$
Graph Separation (d-Separation)

Is there a set $A$ such that the separation statement $(W \perp\!\!\!\!\!\!\perp Y \mid A)$ holds?

Path 1: $W \rightarrow X \rightarrow Z \rightarrow Y$

Path 2: $W \leftarrow T \rightarrow Z \rightarrow Y$

Path 3: $W \rightarrow X \leftrightarrow Y$

$\{X\}$ or $\{Z\}$ or $\{X, Z\}$

$\{T\}$ or $\{Z\}$ or $\{T, Z\}$
Graph Separation (d-Separation)

Is there a set $A$ such that the separation statement $(W \perp Y \mid A)$ holds?

Path 1: $W \rightarrow X \rightarrow Z \rightarrow Y$

Path 2: $W \leftarrow T \rightarrow Z \rightarrow Y$

Path 3: $W \rightarrow X \leftrightarrow Y$

For each path, the set $A$ that satisfies the separation statement is indicated:
- Path 1: $\{X\}$ or $\{Z\}$ or $\{X, Z\}$
- Path 2: $\{T\}$ or $\{Z\}$ or $\{T, Z\}$
- Path 3: not $X$
Is there a set \( A \) such that the separation statement
\[
(W \perp Y \mid A)
\]
holds?

Path 1: \( W \rightarrow X \rightarrow Z \rightarrow Y \)

Path 2:
\[
W \leftarrow T \rightarrow Z \rightarrow Y
\]

Path 3:
\[
W \rightarrow X \leftrightarrow Y
\]
Graph Separation (d-Separation)

Is there a set $\mathcal{A}$ such that the separation statement $(W \perp Y \mid \mathcal{A})$ holds?

Path 1: $W \rightarrow X \rightarrow Z \rightarrow Y$

Path 2: $W \leftarrow T \rightarrow Z \rightarrow Y$

Path 3: $W \rightarrow X \leftrightarrow Y$

Path 4: $W \leftarrow T \rightarrow Z \leftarrow X \leftrightarrow Y$

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Graph Separation (d-Separation)

Is there a set \( A \) such that the separation statement \((W \perp \!\!\!\perp Y \mid A)\) holds?

Path 1: \( W \rightarrow X \rightarrow Z \rightarrow Y \)
Path 2: \( W \leftarrow T \rightarrow Z \rightarrow Y \)
Path 3: \( W \rightarrow X \leftrightarrow Y \)
Path 4: \( W \leftarrow T \rightarrow Z \leftarrow X \leftrightarrow Y \)
Graph Separation (d-Separation)

Is there a set $A$ such that the separation statement 
$(W \perp Y \mid A)$ holds?

Path 1: $W \rightarrow X \rightarrow Z \rightarrow Y$

Path 2: $W \leftarrow T \rightarrow Z \rightarrow Y$

Path 3: $W \rightarrow X \leftrightarrow Y$

Path 4: $W \leftarrow T \rightarrow Z \leftarrow X \leftrightarrow Y$

Winter 2023
Graph Separation (d-Separation)

Is there a set $A$ such that the separation statement $(W \perp Y \mid A)$ holds?

Path 1: $W \rightarrow X \rightarrow Z \rightarrow Y$
Path 2: $W \leftarrow T \rightarrow Z \rightarrow Y$
Path 3: $W \rightarrow X \leftrightarrow Y$
Path 4: $W \leftarrow T \rightarrow Z \leftarrow X \leftrightarrow Y$
Graph Separation (d-Separation)

Is there a set $A$ such that the separation statement $(W \perp Y \mid A)$ holds?

Path 1: $W \rightarrow X \rightarrow Z \rightarrow Y$
Path 2: $W \leftarrow T \rightarrow Z \rightarrow Y$
Path 3: $W \rightarrow X \leftrightarrow Y$
Path 4: $W \leftarrow T \rightarrow Z \leftarrow X \leftrightarrow Y$

Does $A = \{T, Z\}$ suffice?
Graph Separation (d-Separation)

Is there a set $A$ such that the separation statement $(W \perp Y \mid A)$ holds?

Path 1: $W \rightarrow X \rightarrow Z \rightarrow Y$

Path 2: $W \leftarrow T \rightarrow Z \rightarrow Y$

Path 3: $W \rightarrow X \leftarrow Y$

Path 4: $W \leftarrow T \rightarrow Z \leftarrow X \leftarrow Y$
Graph Separation (d-Separation)

Is there a set $A$ such that the separation statement $(W \perp Y \mid A)$ holds?

Path 1: $W \rightarrow X \rightarrow Z \rightarrow Y$

Path 2: $W \leftarrow T \rightarrow Z \rightarrow Y$

Path 3: $W \rightarrow X \leftrightarrow Y \rightarrow Z$

Path 4: $W \leftarrow T \rightarrow Z \leftarrow X \leftarrow Y$

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Graph Separation (d-Separation)

Is there a set $A$ such that the separation statement $(W \perp \!\!\!\!\!\!\perp Y \mid A)$ holds?

Path 1: $W \rightarrow X \rightarrow Z \rightarrow Y$

Path 2:
$W \leftarrow T \rightarrow Z \rightarrow Y$

Path 3: $W \rightarrow X \leftrightarrow Y \rightarrow Z$

Path 4:
$W \leftarrow T \rightarrow Z \leftarrow X \leftrightarrow Y$
Graph Separation (d-Separation)

Is there a set $A$ such that the separation statement $(W \perp Y \mid A)$ holds?

Path 1: $W \rightarrow X \rightarrow Z \rightarrow Y$

Path 2: $W \leftarrow T \rightarrow Z \rightarrow Y$

Path 3: $W \rightarrow X \leftrightarrow Y \leftarrow Z$

Path 4: $W \leftarrow T \rightarrow Z \leftarrow X \leftrightarrow Y$

Winter 2023
Graph Separation (d-Separation)

Is there a set $A$ such that the separation statement $(W \perp Y \mid A)$ holds?

Path 1: $W \rightarrow X \rightarrow Z \rightarrow Y$

Path 2: $W \leftarrow T \rightarrow Z \rightarrow Y$

Path 3: $W \rightarrow X \leftrightarrow Y \rightarrow Z$

Path 4: $W \leftarrow T \rightarrow Z \leftarrow X \leftrightarrow Y$

No such $A$!

Don’t forget the descendants of the colliders!
**d-SEPARATION (EXAMPLE)**

![Diagram](image)

**Figure 1.3:** Graphs illustrating *d*-separation. In (a), $X$ and $Y$ are *d*-separated given $Z_2$ and *d*-connected given $Z_1$. In (b), $X$ and $Y$ cannot be *d*-separated by any set of nodes.