Submodel Decomposition Bound for Influence Diagrams

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35th AAAI Conference on Artificial Intelligence
Outline

• Backgrounds
  • Influence Diagrams (IDs) and Limited Memory IDs (LIMIDs)
  • Decomposition of IDs and LIMIDs
  • Bounding Schemes for Maximum Expected Utility (MEU)

• Submodel Decomposition for IDs and LIMIDs
  • Motivation and Contributions
  • A Submodel-Tree Clustering Scheme
  • A Bounding Scheme over Submodel-Tree

• Experiments and Case Study
  • Upper bounds in IDs, LIMIDs, and MDP/POMDP planning domain

• Conclusion and Future Directions
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Sequential Decision Making Under Uncertainty

\[ R(s_t, a_t) \]
\[ \mathbb{E}[\sum_{t=1}^{T} r_t] \]
\[ \Delta(a_4|a_3, o_3) \]

agent

\[ \mathbb{P}(s_{t+1}|s_t, a_t) \] stochastic dynamics over factored state variables
\[ \mathbb{P}(o_t|s_t, a_{t-1}) \] stochastic partial observation
\[ \Delta(a_4|a_3, o_3) \] stochastic, non-stationary, limited memory policy

system
Influence Diagrams

\[ \mathcal{M} := \langle X, D, P, U, \mathcal{O} \rangle \]

**Chance variables**  
\[ X = \{X_1, X_2, \ldots, X_n\} \]

**Decision variables**  
\[ D = \{D_1, D_2, \ldots, D_m\} \]

**Probability functions**  
\[ P = \{P_1, P_2, \ldots, P_n\} \]

**Utility functions**  
\[ U = \{U_1, U_2, \ldots, U_r\} \]

**Policy functions**  
\[ \Delta = \{\Delta_1, \ldots, \Delta_m\} \]

\[ \mathcal{O} = \{\text{pa}(D_1) \prec D_1 \prec \cdots \prec \text{pa}(D_m) \prec D_m\} \]

\[ \Delta_i(D_i|\text{hist}(D_i)) \]

Maximum expected Utility  
\[ \max_\Delta \mathbb{E}_{P(X,D)} \left[ \sum_{U_i \in U} U_i \right] \]

Optimal strategy  
\[ \Delta^* = \operatorname{argmax}_\Delta \mathbb{E} \left[ \sum_{U_i \in U} U_i \right] \]
\[ \mathcal{M} := \langle X, D, P, U, \mathcal{O} \rangle \]

Policy functions \[ \Delta = \{ \Delta_1, \ldots, \Delta_m \} \]

\[ \Delta_i(D_i|\text{pa}(D_i)) \]

[Lauritzen and Nilsson, 2001]
Graphical Models

$\mathcal{M} := \langle X, D, F \rangle$

Variables $X = \{X_1, X_2, \ldots, X_n\}$
Domains $D = \{D_1, D_2, \ldots, D_n\}$
Functions $F = \{F_1, F_2, \ldots, F_r\}$

Global Function $F(X) = \prod_{F_i \in F} F_i(X_{F_i})$
combination operator: $\otimes, \times, +, \bigotimes$

Inference Task $Z = \sum_X \prod_{F_i \in F} F_i(X_{F_i})$
elimination operator: $\sum$, max, min

Complexity $(\max |D_i|)^\text{tree-width}$ [Decther, 1999]

Bayesian Networks [Pearl 1998]

Primal Graph

Join-Tree

$P(S, C, B, X, D) = P(S) P(C|S) P(B|S) P(X|C,S) P(D|C,B)$

$\text{Primal Graph}$
Decomposition of IDs with Perfect Recall

- Constrained Junction-Tree for IDs [Jensen, 1994]
  - Transform influence diagram to primal graph
  - Use restricted elimination order to obtain constrained tree decomposition

- Decomposition of IDs [Nielsen and Jensen, 1999] [Nielsen, 2001]
  - Identify requisite observation in IDs
  - Extract required subset of variables and functions for each decision variable

- MC-DAG for IDs [Pralet, et. al. 2006]
  - Re-write MEU expression and identify the most relaxed variable elimination order for computing MEU
Decomposition of LIMIDs

- Soluble LIMIDs [Zhang and Poole, 1992] [Lauritzen and Nilsson, 2001]
  - Identify a subclass of LIMIDs that can be solved by variable elimination
  - Local search algorithm that improves single policy function at each iteration

- Local Search for LIMIDs [Detwarasiti and Shacter, 2005] [Maua, 2016]
  - Improve multiple policy functions at each iteration
  - Identify relevant subset of nodes for updating multiple policy functions
Upper bounds for MEU in IDs

- IDs with perfect recall
  - Information Relaxation [Nielsen and Hohle, 2001] [Yuan, et. al. 2010]
  - Join-Graph Decomposition Bounds [Lee, et. al. 2018]
  - Weighted Mini-bucket Decomposition Bounds [Lee, et. al. 2019]

- LIMIDs [Maua and Cozman, 2016]
  - Theoretical Bounds

- Translating IDs with perfect recall
  - Marginal MAP [Liu and Ihler, 2012] [Maua, 2016]
  - MILP Encodings [Parmentier et. al, 2020]
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• Conclusion and Future Work
Motivations and Contributions

• Graph-based method for decomposing IDs and LIMIDs
  • Remove some restrictions in earlier approaches
    • 1 decision per 1 time step, regularity condition, perfect recall

  • Extend tree clustering framework for reasoning in graphical models
    • Identify subproblems from graph
    • Extract a cluster tree for exact algorithms
    • Characterize complexity

• Upper-Bounds for MEU in IDs and LIMIDs
  • Don’t inflate problem size by translation
  • Avoid difficult non-convex optimization formulations in earlier works
Partial Evaluation and Local MEU

• (Definition) Local Maximum Conditional Expected Utility

\[ \text{LMEU}_{M(D', u')} := \max_{\Delta'} \mathbb{E} \left[ \sum_{U_i \in u'} U_i | \text{pa}(D') \right] \]

\[ \max_{\Delta(D_2|C_3,C_4), \Delta(D_3|C_6)} \sum_{X,D} \frac{P(X,D)}{P(C_3,C_4,C_6)} \left[ U_2 + U_3 \right] \]
(Definition) Submodel $\mathcal{M}'(\mathbf{D}', \mathbf{U}')$ is a relevant subset of model $\mathcal{M}$ for computing LMEU on $\mathbf{D}' \subseteq \mathbf{D}, \mathbf{U}' \subseteq \mathbf{U}$

**Submodel**

- **Relevant Observed Variables** $\text{REL}_O(\mathbf{D}', \mathbf{U}')$
- **Relevant Hidden Variables** $\text{REL}_H(\mathbf{D}', \mathbf{U}')$
Stable Submodel

• (Definition) Submodel $\mathcal{M}'(D', U')$ is stable when there is no decision variables in $\text{REL}_H(D', U')$

Unstable Submodel $\mathcal{M}'(\{D_3\}, \{U_3\})$

Stable Submodel $\mathcal{M}'(\{D_2, D_3\}, \{U_2, U_3\})$
Graph-based Identification of Submodels

- $\text{REL}_U(D')$ is descendant utility nodes of decision nodes [Nielsen and Jensen 1999]
Graph-based Identification of Submodels

- $\operatorname{REL}_O(D', U')$ is the backdoor* set between $D'$ and $U'$

(Backdoor) [Pearl 2009]
a set $Z$ satisfies the backdoor criterion relative to $(X, Y)$
1. None of the nodes in $Z$ is a descendant of $X$
2. $Z$ blocks every path between $X$ and $Y$ that contain arrow into $X$

\{C3, C6\} is a backdoor set relative to \{\{D2, D3\}, \{U2, U3\}\}

Removing C3 opens a backdoor path by $C1 \rightarrow C3 \rightarrow D2 \rightarrow C3 \rightarrow C5 \rightarrow U3$
Graph-based Identification of Submodels

- $\text{REL}_H(D', U')$ is the union of all frontdoor* set between $\text{pa}(D')$ and $\text{ch}(U')$

(Frontdoor) [Pearl 2009]
a set $Z$ satisfies the frontdoor criterion relative to $(X, Y)$
1. $Z$ intercept all directed paths from $X$ to $Y$
2. There is no backdoor path from $X$ to $Z$
3. All backdoor paths from $Z$ to $Y$ are blocked by $X$

C1, C2, D1, and C4 don’t belong to any frontdoor set
Submodel–Tree Clustering

- Process decision nodes in reverse topological order

  Partial decision order $\mathcal{O}_D = \{D_1 \prec D_2 \prec D_3\}$

  Process decision variables in the order of D3, D2, and D1

Submodel $\mathcal{M}'(\{D_3\}, \{U_3\})$ is unstable
Submodel–Tree Clustering

- Find a stable submodel

Next combine two submodels \( M'(\{D_3\}, \{U_3\}) \otimes M'(\{D_2\}, \{U_2, U_3\}) \)

and Try \( M'(\{D_2, D_3\}, \{U_2, U_3\}) \)

\( M'(\{D_2, D_3\}, \{U_2, U_3\}) \) is stable
Submodel–Tree Clustering

- Eliminate submodel $\mathcal{M}'(\{D_2, D_3\}, \{U_2, U_3\})$ from IDs $\mathcal{M}$

Remove D2, D3, U2, U3 and Add V(C3)

Remove barren chance nodes C4, C5, C6
Submodel–Tree Clustering

- Identify the next submodel and find a submodel-tree

\[ \mathcal{M}'(\{D_1\}, \{U_1, V\}) \]

Submodel Cluster is a single-stage ID
Submodel Cluster Propagates Conditional MEU
Valuation Algebra over Stable Submodels

• Given an ID $\mathcal{M}$, $\Upsilon_{\mathcal{M}} = \langle \mathcal{M}_{\mathcal{O}_D}, \mathcal{D}_{\mathcal{O}_D}, \otimes, \downarrow \rangle$ is a valuation algebra

  $\mathcal{O}_D$ Partial decision order read from ID

  $\mathcal{M}_{\mathcal{O}_D}$ A set of stable submodels in $\mathcal{M}$ subject to $\mathcal{O}_D$

  $\otimes$ Combination operator for a submodel

  $\downarrow$ Projection operator for a submodel

  $\mathcal{M}_{\mathcal{O}_D}$ A closure of $\mathcal{M}_{\mathcal{O}_D}$ under the combination

  $\text{dom}(\mathcal{M}')$ Domain of a submodel (all variables in $\mathcal{M}'$)

  $\mathcal{D}_{\mathcal{O}_D}$ A set of domains of submodels in $\mathcal{M}_{\mathcal{O}_D}$

[Shenoy 1997] [Kohlas and Shenoy, 2000]
Valuation Algebra over Stable Submodels

• Given an ID $\mathcal{M}$, $\Upsilon_{\mathcal{M}} = \langle M_{O_D}, D_{O_D}, \otimes, \downarrow \rangle$ is a valuation algebra [Kohlas and Shenoy, 2000]

  Semi-group of submodels: $M_{O_D}$ is a semi-group with the combination operation

  Domain of combination: $\text{dom}(M'_1 \otimes M'_2) = \text{dom}(M'_1) \cup \text{dom}(M'_2)$

  Marginalization: $\downarrow_{X} M' = \downarrow_{X \cap \text{dom}(M')} M'$

  $\text{dom}(\downarrow_{X} M') = X \cap \text{dom}(M')$

  $\downarrow_{\text{dom}(M')} M' = M'$

  Transitivity of marginalization: $\downarrow_{X} (\downarrow_{Y} M') = \downarrow_{X \cap Y} M'$

  Distributivity of marginalization over combination: $\downarrow_{X} (M'_1 \otimes M'_2) = M'_1 \otimes (\downarrow_{X} M'_2)$

  Neutral elements: $M'_{0(X)} \otimes M'_{0(Y)} = M'_{0(X \cup Y)}$

• Valuation algebra satisfies axioms of local computation [Shenoy 1997]
Submodel–Tree Decomposition

• Given an ID $M$, and the set of stable submodels $M_{OD}$ relative to $OD$, submodel-tree decomposition is a tuple $T_{ST} := \langle T(C, S), \chi, \psi \rangle$

  $T(C, S)$: Tree of submodel cluster nodes $C$ and separator edges $S$

  $\chi : C \rightarrow 2^{\text{dom}(M)}$: Label a cluster with a subset of variables in $M$

  $\psi : C \rightarrow 2^{M_{OD}}$: Label a cluster with a subset of submodels in $M_{OD}$

  Tree-decomposition satisfies running intersection property

![Diagram of submodel-tree decomposition]

$M'(\{D_1\}, \{U_1, V\})$ $M'(\{D_2, D_3\}, \{U_2, U_3\})$
Submodel–Tree Decomposition

• Minimal submodel-tree decomposition

A submodel-tree decomposition is minimal if submodels assigned at each cluster is not a combination of two stable submodels

• Given an ID, minimal submodel-tree decomposition is unique.

• For IDs with perfect recall, the minimal submodel-tree is equivalent to MC-DAG

• For IDs with perfect recall, each submodel cluster is one time-step ID

• For IDs without perfect recall, each submodel cluster defines the scope of exhaustive search
Message Passing over a Submodel-Tree

\[ M'(\{D_1\}, \{U_1, V\}) \]

\[ M'(\{D_2, D_3\}, \{U_2, U_3\}) \]

\[ V(C_3) = \max_{\Delta(D_2, D_3|C_3, C_6)} \mathbb{E}[U_2 + U_3|C_3] \]

\[ P(C_3) = \sum_{C_1, C_2, D_1} P(C_1, C_2, C_3, D_1) \]

- Each submodel can be solved by any exact algorithm for propagating messages
The time and space complexity for solving IDs over the submodel-tree decomposition is exponential in submodel-tree width \( w_s : \max_{C \in C} w_c(C') \), where \( w_c(C) \) is the constrained tree-width of the submodel at \( C \).
Bounding MEU of Each Submodel

• Exponentiated Utility Bounds for MEU

For each submodel cluster, we can apply Jensen’s inequality to bound MEU

\[
\max_{\Delta} \mathbb{E} \left[ \sum_{U_i \in U} U_i(X_{U_i}) \right] \leq \max_{\Delta} \log \mathbb{E} \left[ e^{\sum_{U_i \in U} U_i(X_{U_i})} \right]
\]

\[
= \log \max_{\Delta} \mathbb{E} \left[ e^{\sum_{U_i \in U} U_i(X_{U_i})} \right]
\]

\[
= \log \max_{\Delta} \mathbb{E} \left[ \prod e^{U_i(X_{U_i})} \right]
\]

LSH: MEU expression with additive utility function
RHS: Upper bound of MEU with log-partition function with exponentiated utility functions

• Use “any” upper bounding scheme for MMAP on RHS
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### Benchmark Domains

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Experiments: Synthetic IDs

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<th>JGDID(i=1)</th>
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- ST-GDD: submodel-tree decomposition + GDD for MMAP [ping et al 2015]
- ST-WMB: submodel-tree decomposition + WMBMAM for MMMAP [marinescu et al 2014]
- JGDID: constrained-join graph + GDD for IDs [Lee et al 2018]
- WMBMEID: constrained mini-bucket tree + WMB/GDD for IDs [Lee et al 2019]
- Evaluation: average of the gap $\frac{U-U_{\text{min}}}{U}$
Experiments: Synthetic LIMIDs

- $|C|$: number of clusters in submodel tree
- Kpu-UB: Analytical bound by [Maua and Cozman 2016]
Experiments: SysAdmin MDP/POMDP [Guestrin, et. al 2003]

- Evaluation
  - UB: WMBMM-EXP (i=20)
  - LB: Online planner to obtain lower bounds

\[
gap = 1 - \frac{LB}{UB}
\]
## SysAdmin MDP

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<th>(p)</th>
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## SysAdmin POMDP

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Conclusion and Future Directions

• Extend Tree-Clustering Framework in PGM for IDs and LIMIDs
  • Graph-based tree-clustering procedure for IDs and LIMIDs
  • Hierarchical message passing algorithm for exact inference

• Simple and Scalable Bounding Scheme for IDs
  • Exponentiating utility functions and reuse decomposition bounds for MMAP

• Future Directions
  • Guide heuristic search for finding MEU in IDs and LIMIDs
  • Extend relaxation schemes in PGM to submodel-tree decomposition
  • Submodel-tree clustering framework for multi-agent IDs