CS 163 & CS 265: Graph Algorithms

Week 2: Spanning trees and DAGs

Lecture 7: Spreadsheets and topological ordering

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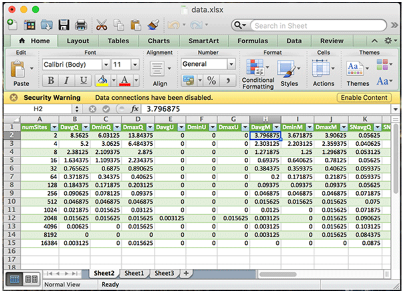
Spreadsheets

Spreadsheets

Examples: Excel, Visicalc, Google sheets



Security Warning Data connections have been disabled.

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	A	B	C	D	E	F	G	H	I	J	K
1	numSites	DavgQ	DminQ	DmaxQ	DavgU	DminU	DmaxU	DavgM	DminM	DmaxM	SNavgQ
2	2	8.5625	6.03125	13.84375	0	0	0	3.796875	3.671875	3.90625	0.05625
3	4	5.2	3.0625	6.484375	0	0	0	2.303125	2.203125	2.359375	0.040625
4	8	2.38125	2.109375	2.875	0	0	0	1.271875	1.25	1.296875	0.053125
5	16	1.634375	1.109375	2.234375	0	0	0	0.69375	0.640625	0.78125	0.05625
6	32	0.765625	0.6875	0.890625	0	0	0	0.384375	0.359375	0.40625	0.059375
7	64	0.371875	0.34375	0.40625	0	0	0	0.2	0.171875	0.21875	0.059375
8	128	0.184375	0.171875	0.203125	0	0	0	0.09375	0.09375	0.09375	0.05625
9	256	0.090625	0.078125	0.09375	0	0	0	0.046875	0.046875	0.046875	0.071875
10	512	0.046875	0.046875	0.046875	0	0	0	0.015625	0.015625	0.015625	0.075
11	1024	0.021875	0.015625	0.03125	0	0	0	0.0125	0	0.015625	0.071875
12	2048	0.015625	0.015625	0.015625	0.003125	0	0.015625	0.003125	0	0.015625	0.090625
13	4096	0.00625	0	0.015625	0	0	0	0.003125	0	0.015625	0.103125
14	8192	0	0	0	0	0	0	0.003125	0	0.015625	0.084375
15	16384	0.003125	0	0.015625	0	0	0	0	0	0	0.0875
16											
17											
18											

Sheet2 Sheet1 Sheet3

Array of cells, each containing some information:

A static string or a number

A formula for computing the cell value from other cells

What happens when you change one cell?

Each formula cell also stores its value,
used when the spreadsheet is displayed or printed

The formula for this value may use values from other cells

When any of those other cells changes,
we need to recompute the formula value

Changes its value, may cause other formulas to be recomputed

But if many formula values need recomputing,
what order should we use?

Top-down left-to-right can be very bad
for spreadsheets where information propagates bottom-up

Natural order recalculation

Choose a special ordering of spreadsheet cells such that:

- If any cell x has a formula that uses the value of another cell y, then x is later than y in the ordering
- Therefore, y's value will have already been recalculated by the time it is needed to recalculate x
- If we can order cells in this way, we can propagate changes to all cells that depend on the changed cell (directly or indirectly) while recalculating each cell at most once

Can be applied either to whole spreadsheet, or to only the subset of cells that need recalculation

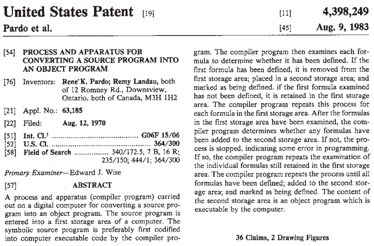
Intellectual property issues

Algorithms for this problem have been known and published since 1963.

Despite that, Pardo and Landau filed for a patent in 1970, and (after a long process) it was granted in 1983

Became an important precedent for the question of whether algorithms can be patented

It was ruled unenforceable in 1995 and has since expired



United States Patent [19] — [11] **4,398,249**

Pardo et al. — [45] **Aug. 9, 1983**

[54] **PROCESS AND APPARATUS FOR CONVERTING A SOURCE PROGRAM INTO AN OBJECT PROGRAM**

[76] Inventors: **Rene'K. Pardo; Remy Landau,** both of 12 Romney Rd., Downsview, Ontario, both of Canada, M3H 1H2

[21] Appl. No.: **63,185**

[22] Filed: **Aug. 12, 1970**

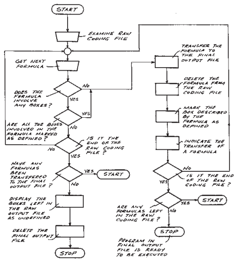
[51] **Int. Cl.**³ **G06F 15/06**
[52] **U.S. Cl.** **364/300**
[58] **Field of Search** .... 340/172.5, 7 B, 16 R; 235/150; 444/1; 364/300

Primary Examiner—Edward J. Wise

[57] **ABSTRACT**

A process and apparatus (compiler program) carried out on a digital computer for converting a source program into an object program. The source program is entered into a first storage area of a computer. The symbolic source program is preferably first codified into computer executable code by the compiler program. The compiler program then examines each formula to determine whether it has been defined. If the first formula has been defined, it is removed from the first storage area; placed in a second storage area; and marked as being defined. if the first formula examined has not been defined, it is retained in the first storage area. The compiler program repeats this process for each formula in the first storage area. After the formulas in the first storage area have been examined, the compiler program determines whether any formulas have been added to the second storage area. If not, the process is stopped, indicating some error in programming. If so, the compiler program repeats the examination of the individual formulas still retained in the first storage area. The compiler program repeats the process until all formulas have been defined; added to the second storage area; and marked as being defined. The content of the second storage area is an object program which is executable by the computer.

36 Claims, 2 Drawing Figures


START
EXAMINE RAW CODING FILE
GET NEXT FORMULA
DOES THE FORMULA INVOLVE ANY BOXES?
No
YES
YES
ARE ALL THE BOXES INVOLVED IN THE FORMULA MARKED AS DEFINED?
No
IS IT THE END OF THE RAW CODING FILE?
No
YES
HAVE ANY FORMULAS BEEN TRANSFERED TO THE FINAL OUTPUT FILE?
YES
START
No
DISPLAY THE BOXES LEFT IN THE RAW OUTPUT FILE AS UNDEFINED
DELETE THE FINAL OUTPUT FILE
STOP
TRANSFER THE FORMULA TO THE FINAL OUTPUT FILE
DELETE THE FORMULA FROM THE RAW CODING FILE
MARK THE BOX DESCRIBED BY THE FORMULA AS DEFINED
INDICATE THE TRANSFER OF A FORMULA
No
IS IT THE END OF THE RAW CODING FILE?
YES
ARE ANY FORMULAS LEFT IN THE RAW CODING FILE?
YES
START
No
STOP
PROGRAM IN FINAL OUTPUT FILE IS READY TO BE EXECUTED

Graph formulation of the natural order problem

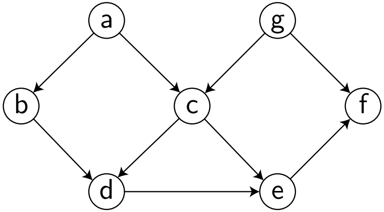
Construct a directed graph from the spreadsheet

- A vertex for every cell
- Edge $x \to y$ when value from x is used in formula for y
  edge direction = direction of information flow

Natural order of cells = list of all vertices so that,
for every edge $x \to y$, x is earlier than y in the list

Example:

- Not valid: a, b, c, d, e, f, g
  Edges g→c and g→f go wrong way, later to earlier
- Valid: a, b, g, c, d, e, f


$$\begin{tikzcd}
 & a \arrow[dl] \arrow[dr] & & g \arrow[dl] \arrow[dr] & \\
b \arrow[dr] & & c \arrow[dl] \arrow[dr] & & f \\
 & d \arrow[rr] & & e \arrow[ur] &
\end{tikzcd}$$

Definitions

Topological order and topological sorting

Given a directed graph G

Topological ordering of G: List of all vertices in G so that, for every edge $x \to y$, x is earlier than y in the list

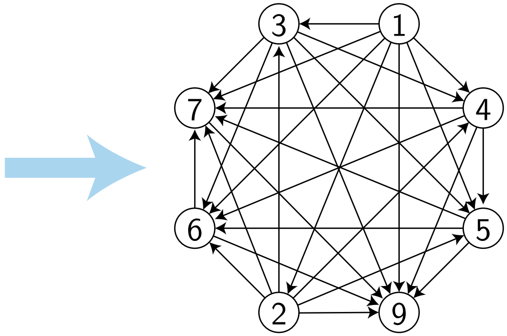
Topological sorting: the problem of computing a topological ordering of a given graph

Key questions: Which graphs have topological orders?
How can we find them quickly?

Why it's called sorting

if you turn a list of numbers into a graph with a vertex for each number and an edge $x \to y$ when $x < y$, then topologically sorting the graph is the same as sorting the numbers

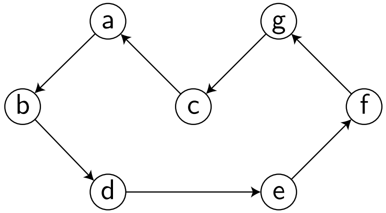
3, 1, 4, 5, 9, 2, 6, 7



Some graphs cannot be ordered

Tour = walk that starts and ends at the same vertex

Cycle = tour with no other repetitions



A graph that contains a cycle has no topological ordering

The vertex from the cycle that is first in the ordering will have an incoming edge from a later vertex

Example: for ordering a,b,c,d,e,f,g, edge c$\rightarrow$a goes the wrong way

Directed acyclic graphs

Theorem: For every directed graph G, either

- G has a topological order, or
- G contains a cycle

(but not both!)

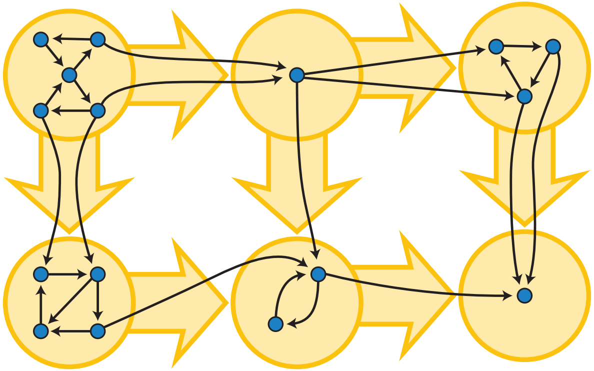
How to prove it: find an algorithm that always finds a cycle or a topological order

In spreadsheet application, raise an error message when a new formula would create a cycle

The directed graphs that do not have cycles (and that do have topological orders) are called directed acyclic graphs

Condensation

When a graph has cycles, collapsing its strongly connected components to supervertices produces a directed acyclic graph



Can be used to extend DAG algorithms to other graphs

Algorithms

Certifying algorithms

Easier to check correctness of a topological order than to find one:

- Label vertices by their position in the order
- Check that all vertices are listed exactly once
- Check that each edge $x \to y$ has $\text{label}(x) < \text{label}(y)$

It's also easy to test that a cycle really is a cycle in the graph

If an algorithm produced only a Boolean answer (there is a cycle) rather than outputting one, it would be harder to check

Some software systems use this as a design principle: whenever possible, produce an easy-to-test answer, and then test it

- If a bug leads to an incorrect answer, discover it quickly
- Because checkers are simpler than the algorithms they check, you can be more confident that their implementation is correct

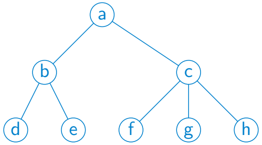
DFS-based topological ordering algorithm

List vertices in reverse of postorder traversal of DFS forest

```
def toporder(G):
    def visit(v):
        add v to reached
        for each edge from v to w:
            if w not in reached:
                visit(w)
        add v to end of postorder

    reached = empty set
    postorder = empty list
    for each vertex v in G:
        if v not in reached:
            visit(v)
    return reverse(postorder)
```

Postorder of a tree:
list children recursively,
then root

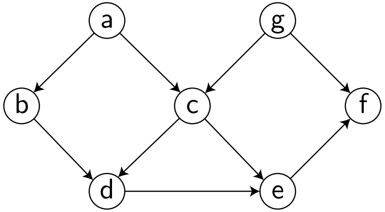


Postorder:
d, e, b, f, g, h, c, a

Reverse:
a, c, h, g, f, b, e, d

Example

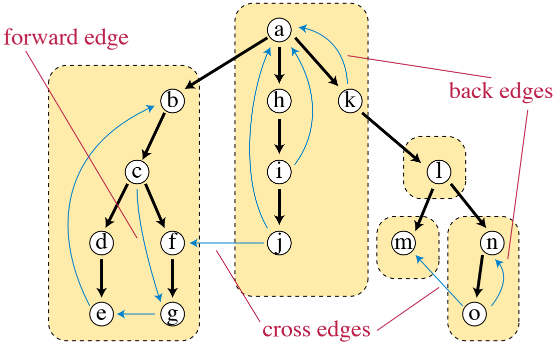
```
visit(a)
  visit(b)
    visit(d)
      visit(e)
        visit(f)
        add f to postorder
      add e to postorder
    add d to postorder
  add b to postorder
  visit(c)
  add c to postorder
add a to postorder
visit(g)
add g to postorder
```



Postorder: f, e, d, b, c, a, g
Reverse: g, a, c, b, d, e, f

Why does this work?

Back to classification of edges in DFS, from strong connectivity:



DFS tree edges, forward edges, and cross edges are all ordered correctly (earlier to later) by reverse postorder

Back edges are not ordered correctly

Back edge + path in tree from ancestor to descendant = cycle

Modifications to DFS to detect and find cycles

- Maintain an “active set” of the ancestors of the current vertex (the vertices still on the call stack); add a vertex to the set when its visit starts, and remove before returning
- Store pointers from each vertex to its parent in the DFS forest
- When we find an edge to a vertex in the active set, it is a back edge and forms a cycle
- Use parent pointers to find the cycle and raise an exception

DFS-based topological ordering + cycle-finding

```
def visit(v):
    add v to active
    add v to reached
    for each edge v to w:
        if w not in reached:
            parent[w] = v
            visit(w)
        else if w in active:
            cycle = [v]
            while v != w:
                v = parent[v]
                cycle.append(v)
            raise cycle
    add v to end of postorder
    remove v from active
```

```
active = empty set
reached = empty set
postorder = empty list
parent = empty dictionary
for each vertex v in G:
    if v not in reached:
        visit(v)
return reverse(postorder)
```

Morals of the story

Every directed graph has either a topological ordering, or a cycle

(usually, many orderings or many cycles)

We can find an ordering or a cycle in linear time using DFS

There may be many orderings or cycles; we only find one

If it has a topological ordering, it's called a directed acyclic graph

Topological orderings are good for recomputing spreadsheet values

We will see other applications

Certifying algorithms: Useful design style