

CS 163 & CS 265: Graph Algorithms

Week 9: Flow

Lecture 24: Definition, properties, and applications

David Eppstein

University of California, Irvine

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Baseball elimination

Background

Pro baseball teams are grouped into divisions

- ▶ Currently six divisions, five teams each
- ▶ Both numbers have varied over time

All teams play same the same total number of games per season, both against teams in their division and others

Baseball games always have a winner and a loser (no ties)

At the end of the season, the team from each division with the most wins goes on to the post-season series of championships leading to the World Series (as do two “wildcard” teams which are not important for this application)

If there is a tie for first place, the tied teams play another tie-breaker game

Elimination

Close to the end of the season, but with a small number of games remaining to play...

Team X **still has a chance** if some scenario for who wins each remaining game in the division (no matter how unlikely) leaves Team X with the most wins or tied for the most

Otherwise, Team X is **mathematically eliminated**

Example:

- ▶ Suppose remaining games are X vs W, and Y vs Z.
- ▶ Suppose also that Y and Z each have one more win than X.
- ▶ X can tie with the loser of the Y–Z game by winning the game against W, but will still be eliminated by the winner of the Y–Z game, so X is mathematically eliminated

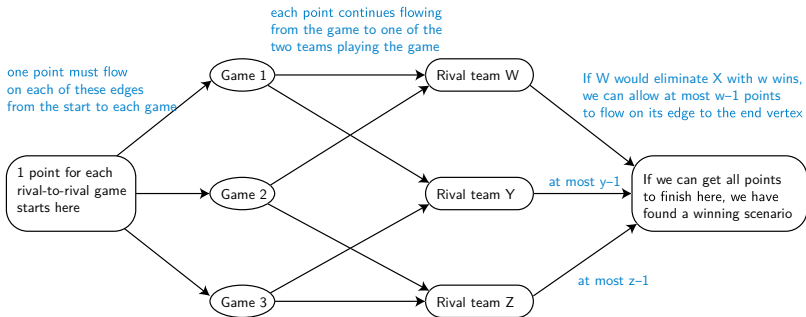
Some simplifying assumptions

If we're trying to test whether team X is eliminated, it's safe to only think about scenarios where X wins all of their remaining games, so we know how many wins team X will end up with

Similarly, if a rival team Y from X 's division has a game against an out-of-division team, it's safe to assume that Y loses (because we're looking for the most favorable scenario for X).

The real question is: for the games between pairs of rival teams Y and Z , both from the same division, is it possible to split the wins so that no rival ends up with too many?

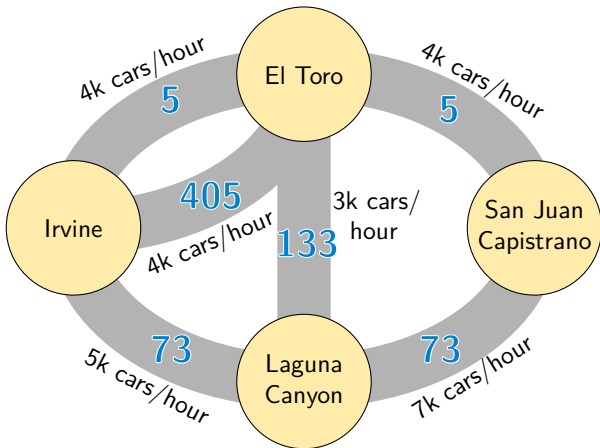
A graphical view



This kind of graph (with quantities that can flow from vertex to vertex, with limits on how much can flow across each edge) is called a **flow network**, and finding a way to get everything from start to finish obeying the limits is a **flow problem**.

Maximum flow

Another application: Traffic engineering

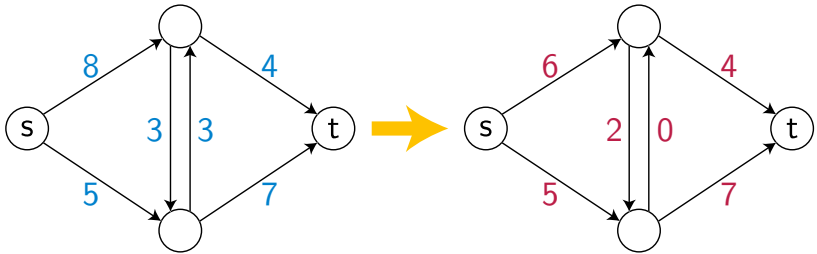


How many cars/hour can get from Irvine to San Juan Capistrano?

Widest path (73) has 5k cars/hour, but we can do much better!

$$5k (73) + 4k (5) + 2k (405+133+73) = 11k$$

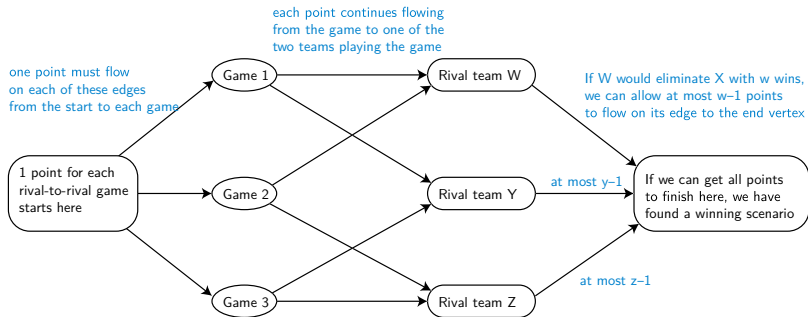
The maximum flow problem



Input: Directed graph, designated start vertex s , end vertex t
Every edge is labeled with a positive number, its **capacity**

Output: **flow amounts** on each edge, $0 \leq \text{flow amount} \leq \text{capacity}$
At vertices other than s and t , total flow in = total flow out
Goal: Maximize flow out of s = flow into t

Baseball elimination as maximum flow



Left and center edges have **capacity = 1**

Right edges have **capacity = # of allowable wins**

Goal: Find a flow with total flow amount = total # of games

Properties

Can be formulated as a linear program

Find flow amounts $f_{i,j}$ for each edge $i \rightarrow j$

Satisfy linear constraints $0 \leq f_{i,j} \leq c_{i,j}$
(where $c_{i,j}$ is the capacity of edge $i \rightarrow j$)

and

$$\sum f_{i,j} = \sum f_{j,k} \text{ for all vertices } j \notin \{s, t\}$$

Maximize linear function $\sum f_{s,i}$

Existence and computability

Even when the capacities are irrational numbers,
there always exists a maximum flow

... rather than an infinite sequence of flows converging to the
maximum but never quite reaching it

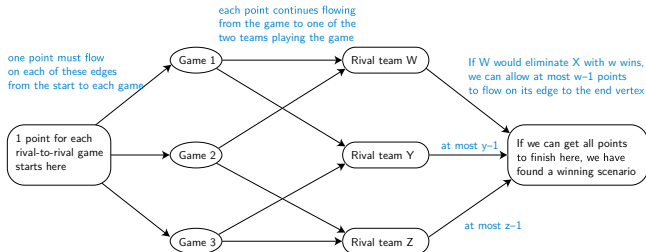
We can find it with an algorithm that runs in polynomial time

Linear programming and some specialized flow algorithms have
running time that depends on the numerical value of the
capacities, but it can also be solved in time that is polynomial in
just n and m , independent of capacity

Integrality

If all the capacities are integers, then there exists a maximum flow all of whose flow amounts are integers

Not true of some other linear programming problems

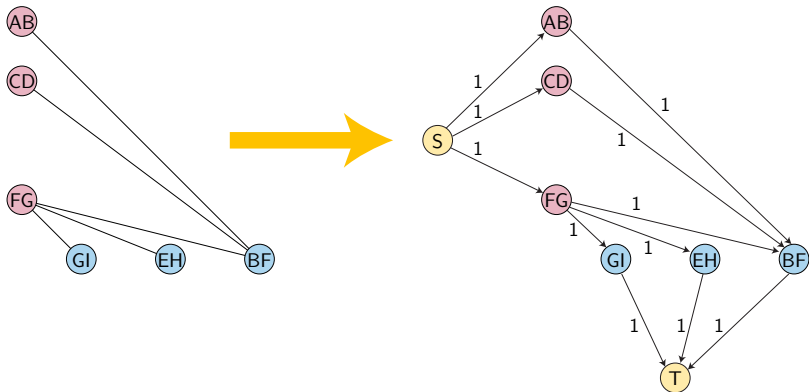


We need this property for baseball elimination, corresponding to the fact that every game has a winner (no ties)

Maximum bipartite matching as flow

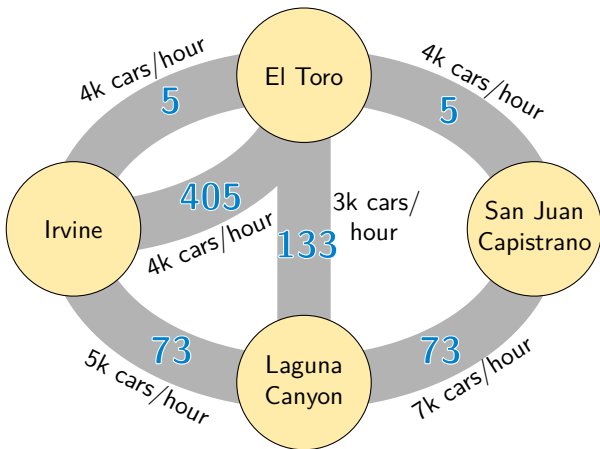
Given a bipartite graph G :

- ▶ Add new vertices s connected to all red vertices, and t connected to all blue vertices
- ▶ Direct every edge $s \rightarrow \text{red} \rightarrow \text{blue} \rightarrow t$, with capacity = 1
- ▶ Find an integer maximum flow; matching = edges with flow 1



Decomposition into paths

We can always decompose the flow into multiple paths

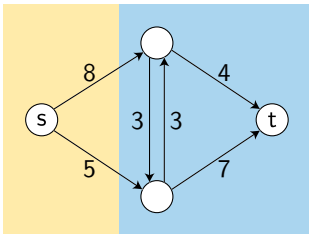


... as we did in the traffic example

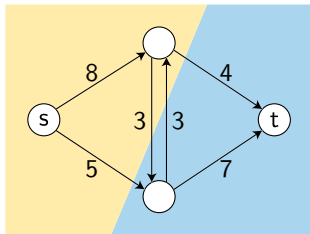
The max-flow min-cut theorem

Cuts

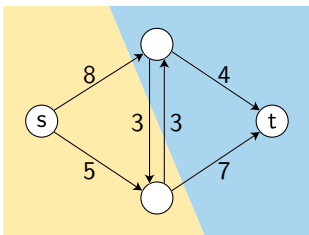
Split vertices into 2 subsets, with s and t in different sets



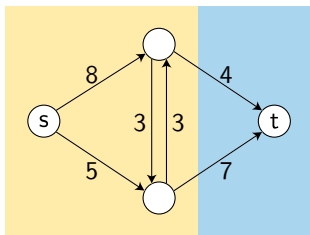
$$\text{capacity} = 8 + 5 = 13$$



$$\text{capacity} = 5 + 3 + 4 = 12$$



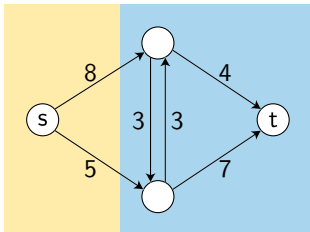
$$\text{capacity} = 8 + 3 + 7 = 18$$



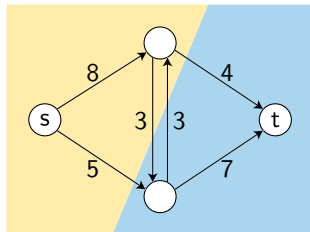
$$\text{capacity} = 4 + 7 = 11$$

Capacity of a cut

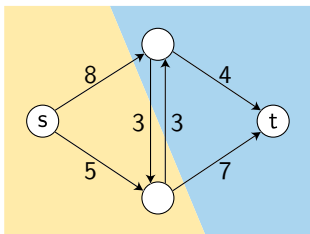
Sum capacities of edges from s 's subset to t 's subset



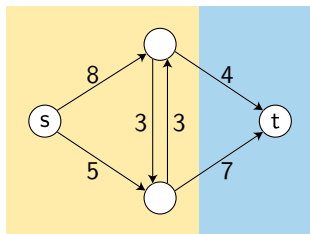
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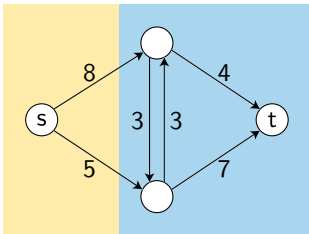
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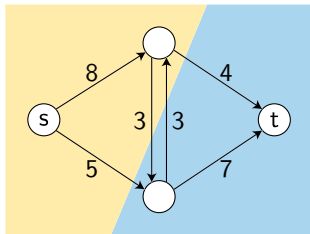
$$\text{capacity} = 4 + 7 = 11$$

Minimum cut problem

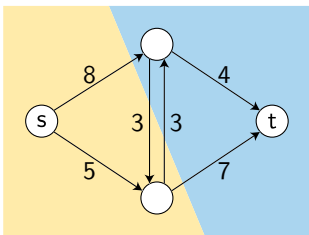
Given a flow network, find cut with smallest capacity



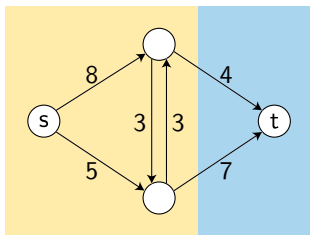
$$\text{capacity} = 8 + 5 = 13$$



$$\text{capacity} = 5 + 3 + 4 = 12$$



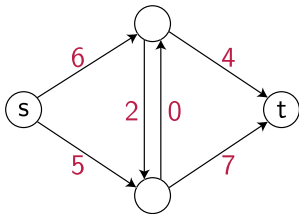
$$\text{capacity} = 8 + 3 + 7 = 18$$



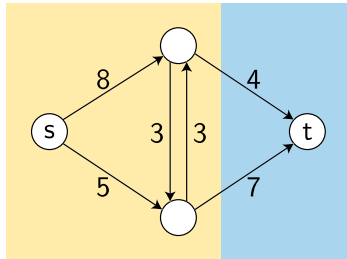
$$\text{capacity} = 4 + 7 = 11$$

The max-flow min-cut theorem

For every flow network, maximum flow = minimum cut



flow out of $s = 6 + 5 = 11$



capacity = $4 + 7 = 11$

max flow \leq min cut is easy (can't get more flow across the cut)

We will show that, for every flow (maximum or not), either there is an equal cut, or we can increase the flow \Rightarrow max flow \geq min cut

Morals of the story

Definition: at all vertices except s and t , flow in equals flow out

Maximum flow: obey capacity constraints, maximize flow out of s

Applications include baseball elimination, traffic, matching

Existence of integer flows for integer capacities

The minimum cut problem and max flow min cut theorem