CS 163 & CS 265: Graph Algorithms

Week 10: Planar graphs

Lecture 10c: Straightening planar drawings

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Overview

Two stages of finding planar drawings

Stage 1: Topological drawing

- No coordinates for vertices
- Describe the ordering of the edges around each vertex
- Allow curved edges
- Wednesday's lecture

Stage 2: Straighten it

- Fáry's theorem: always possible [Wagner 1936; Fáry 1948; Stein 1951]
- Give coordinates for the vertices (small integers)
- Draw edges as straight line segments
- Today

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{ 0: [1,2,3],
  1: [0,2,3],
  2: [0,1,3],
  3: [0,1,2] }
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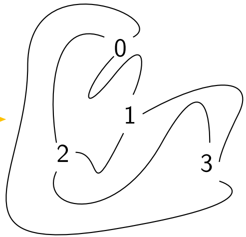
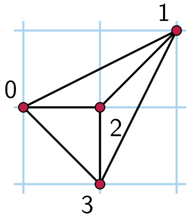
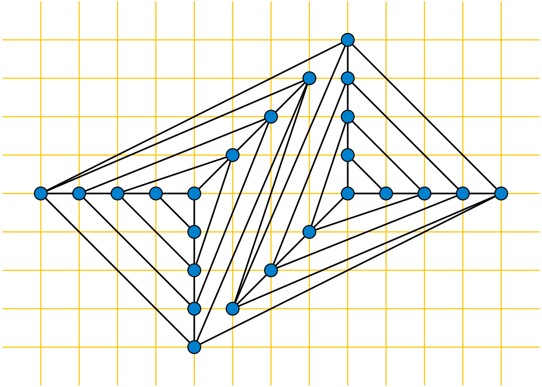




Grid drawing lower bound

Drawing an n-vertex graph may require an $n/3 \times n/3$ grid
(each level of nested triangles adds to the size of the grid)



[Dolev et al. 1984]

Grid drawing upper bounds

Every planar graph can be drawn with its vertices in:

- A triangular subset of a $(2n-3) \times (n-1)$ grid [de Fraysseix et al. 1990]
- A triangular subset of an $(n-1) \times (n-1)$ grid [Schnyder 1990]
- A triangular subset of an $\frac{4n}{3} \times \frac{2n}{3}$ grid [Brandenburg 2008]

For today's lecture we'll follow Schnyder's method.

Maximal planar graphs

Maximal planar graphs

A planar graph is maximal if you cannot add any more edges while preserving planarity

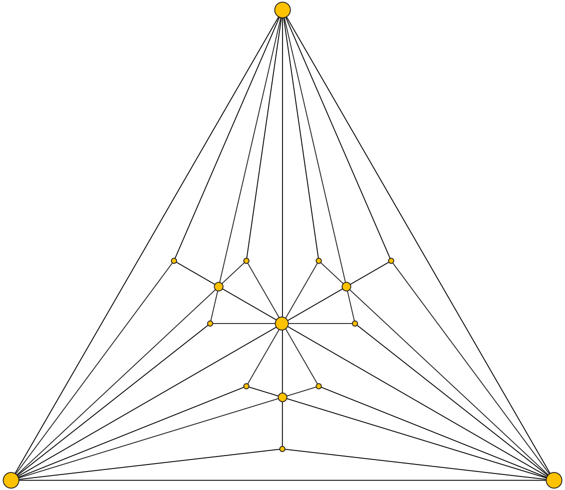
If it has a face with four or more vertices, it cannot be maximal

- Suppose the consecutive vertices are a, b, c, and d
- If any pair of them is not adjacent, we can add a new edge connecting them inside the face
- Otherwise, we could get the graph K_5 by adding a new vertex e inside the face, adjacent to all four $\Rightarrow$ impossible for a planar graph

If there is no face with four or more vertices, $m = 3n - 6$
$\Rightarrow$ we have reached the maximum # of edges

Maximal = all faces are triangles

(including the outer face)



Making a planar graph maximal

If we want to draw a planar graph with straight edges, it doesn't hurt to add extra edges

Find a planar embedding

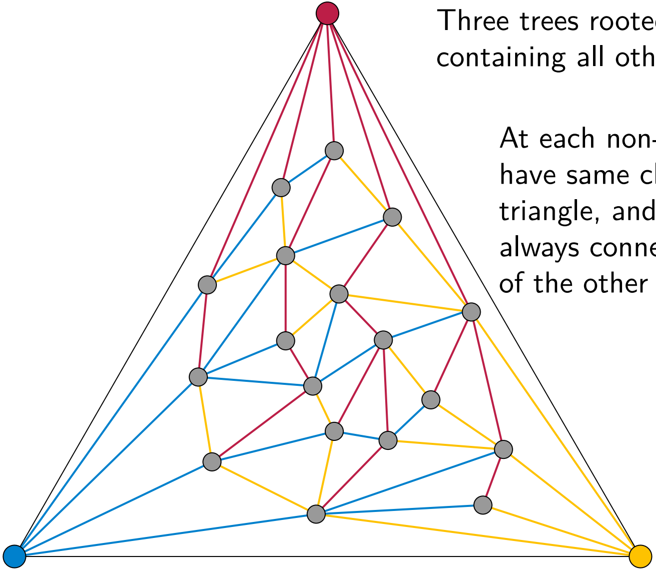
While there is any face with four or more vertices:

- Let a, b, c, and d be consecutive on the face
- Find two non-adjacent vertices among these four
- Add the edge between these two vertices, splitting the face into two faces

Can be done in linear time

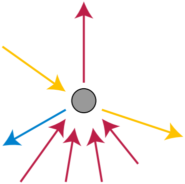
Schnyder forests

Schnyder forest



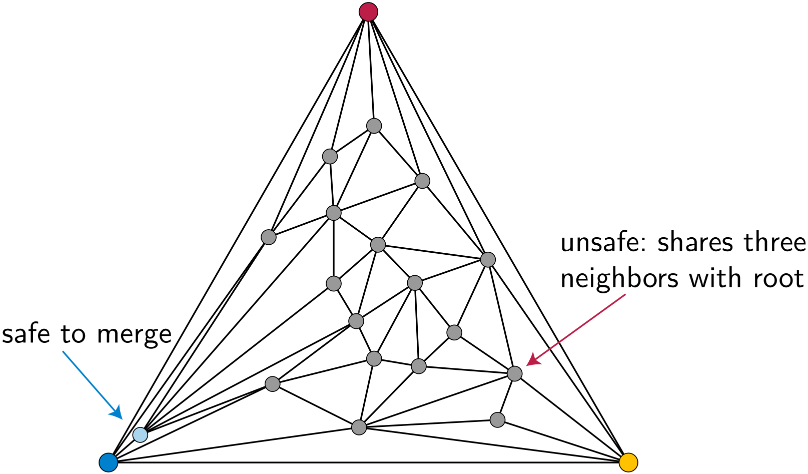
Three trees rooted at the outer vertices, containing all other vertices of the graph

At each non-root, edges to parents have same clockwise order as outer triangle, and children from one tree always connect between the parents of the other two trees



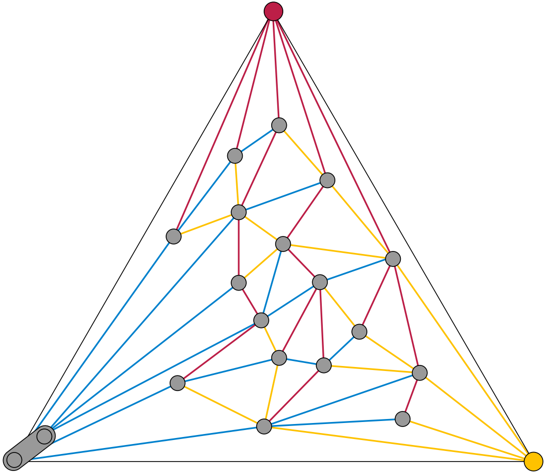
Construction algorithm, step 1

Find a neighbor of one of the three root vertices that can be safely merged to the root (has only two shared neighbors with the root)



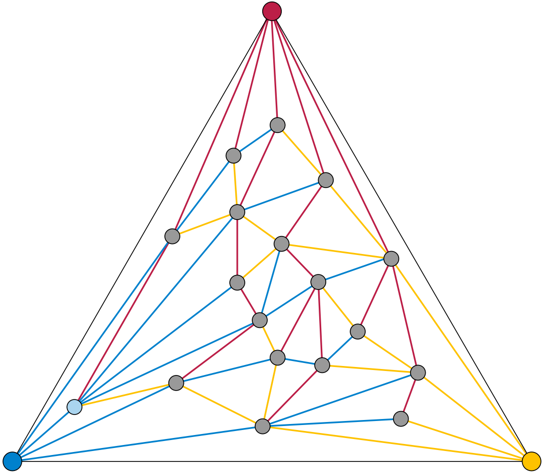
Construction algorithm, step 2

Recursively find a Schnyder forest for the smaller graph



Construction algorithm, step 3

Unmerge and color: Use the other two colors for its edges to neighbors of the root, and the root color for all other edges

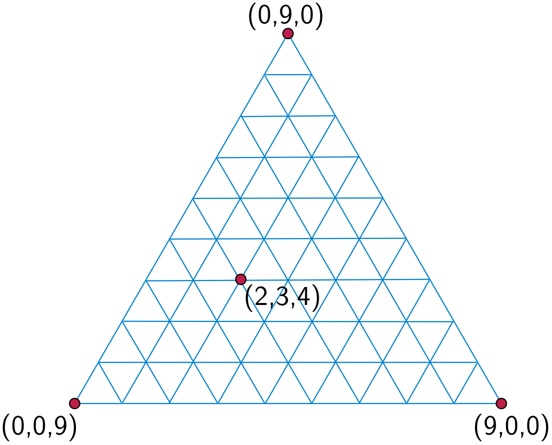


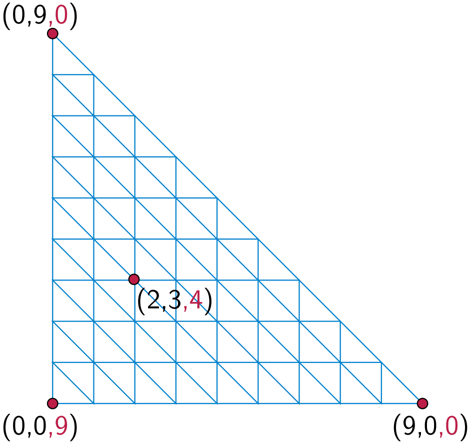
Coordinates

Trilinear coordinates

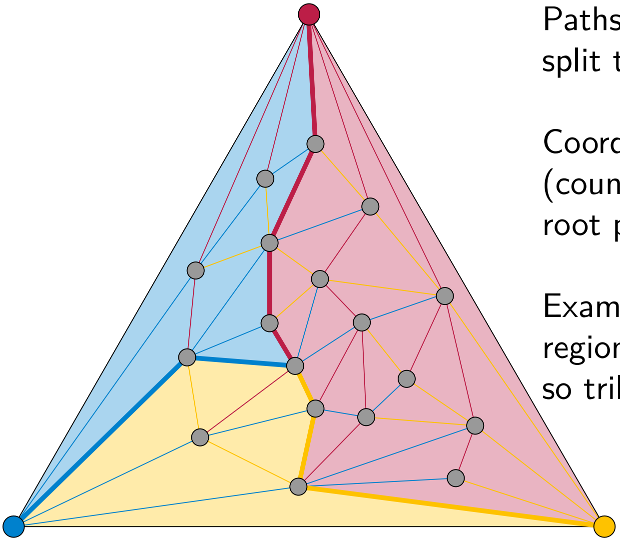
In an equilateral triangle, parameterize each point by three numbers, its distance from the three sides

All points have equal sums of coordinates





Numbers from Schnyder forests

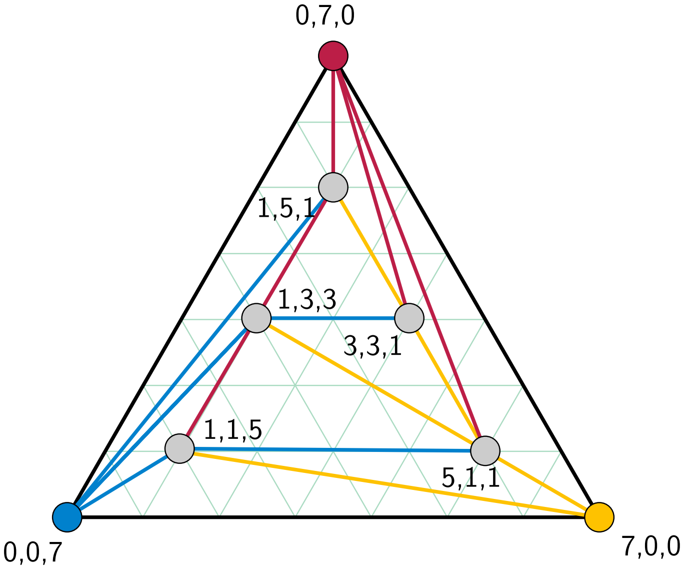


Paths from v to the 3 tree roots split the graph into 3 regions

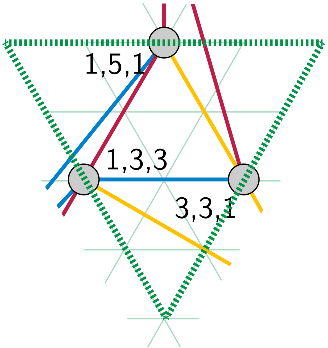
Coordinates(v) = #vertices/region (counting vertices on same-color root path, but not v itself)

Example vertex has 4 points in blue region+path, 4 in yellow, 12 in red, so trilinear coordinates = (4,4,12)

A smaller example



Why does it work?



Inscribe each face of drawing on an upside-down equilateral triangle

Vertices appear in correct clockwise order $\Rightarrow$ geometric orientation is same as topological orientation

If drawing does not fold up on itself anywhere locally, it is planar globally

The morals of the story

We can find topological planar drawings (previous lecture)

We can add edges to the drawing to make all faces triangles

We can partition the edges into the three trees of a Schnyder forest

Counting vertices in regions gives coordinates for a straight-line drawing in a small grid

All in linear time

References I

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References II

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