CS 163 & CS 265: Graph Algorithms
Week 1: Introduction
Lecture 1c: Reachability, DFS, and BFS

David Eppstein
University of California, Irvine

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Reachability and implicit graphs
We’ve discussed how web search engines build indexes of web pages and sort the results of queries, but how did they find those pages in order to build an index of them?

**Web crawler or spider**

Algorithm that explores the web graph or any unknown graph finding all of the vertices that it can reach by following links from a given starting vertex

Example: find all web pages on the www.ics.uci.edu web site that can be reached from the starting page https://www.ics.uci.edu/

Pages that are not linked from other pages will not be found
Web graph: We do not already have its vertices and edges in our computer (that’s the problem we’re trying to solve) but they still exist out on the internet.

We can represent any vertex by its url, a string.

Given any vertex we can find its outgoing edges by reading and parsing the html content at that url.

More generally, an implicit graph is any graph that is not already represented in our computer, where we have some way of describing vertices and finding their edges.
Another application of implicit graphs: Puzzles

Vertices = states of the puzzle, edges = moves

Some puzzles like Rubik’s cube have many more states than we can store in computer memory at once

Real-world application in planning sequences of actions
Paths and walks

**Path** in an undirected graph: alternating sequence of vertices and edges, without repetitions, starting and ending at vertices, with each edge sandwiched by its two endpoints.

In a directed graph: same, but each edge should have starting endpoint before it and destination endpoint after it in the sequence, so that the edges are all oriented consistently along the path.

![Graphs with paths and walks](image)

**Walk**: allow repeated vertices and edges.

Unfortunately some sources use different terminology: “simple path” for no repetitions, or “path” allowing repetitions.

A single vertex by itself is a path (very short alternating sequence)
Reachability

Vertex \( v \) can reach vertex \( w \) in graph \( G \)

\[ \iff G \text{ has a path starting at } v \text{ and ending at } w \]

\[ \iff G \text{ has a walk starting at } v \text{ and ending at } w \]

Repetitions in any walk can be shortcut to make a path

Every vertex can reach itself by a one-vertex path

Reachability is transitive: if \( G \) has paths \( u \leadsto v \) and \( v \leadsto w \), we can connect them together to form a walk \( u \leadsto w \)

In directed graphs, it can be asymmetric: there might exist a path \( u \leadsto v \) but not one in the reverse direction

In undirected graphs, it is symmetric, any path \( u \leadsto v \) can be reversed to give a path \( v \leadsto u \)
Formulation of the web spider problem

Given an implicit graph $G$ and a starting vertex $s$

Find and return the set of all vertices that can be reached from $s$

Algorithm doesn’t have to know that the graph is implicit, but in the web spider application some graph operations are limited: for instance we cannot easily find incoming edges to vertices

Web spiders don’t need to return the paths from $s$ to the reachable vertices but other applications might need them (e.g. for puzzles, paths are sequences of moves that solve them)
Depth-first search
Depth-first search

Easy recursive exploration method

- Main idea: Whenever you find a new reachable vertex $v$, recursively explore its neighbors, the vertices that can be reached in one more step from $v$
- Start by exploring the starting vertex $s$ and its neighbors
- Very important optimization: keep track of the vertices you’ve already explored and don’t re-explore if you reach them again. Otherwise infinite loops are possible
- The set of already-explored vertices can be re-used as the return value from the search
def reachable(G, s):
    // arguments are an implicit graph G
    // and the starting vertex s

    reached = set()

    def recurse(v):
        add v to reached
        for each edge in G from v to w:
            if w is not already in reached:
                recurse(w)

    recurse(s)
    return reached
Depth-first reachability example

Tree in bottom shows recursive calls, but it is also a tree in the graph, with paths from \( s \) to all other vertices. It is called the DFS tree.

The algorithm can be modified to construct and return it.
Analysis of depth-first search

- We have at most one call to recurse per vertex, because after that it is in the reached set and won’t be called again.
- Inside each recursive call there is a loop over its neighbors.
- Naive analysis: $n$ calls $\times n$ iterations of the loop $= O(n^2)$
  but we can do much better!

- In a directed graph, each edge is looped over at most once (in the recursive visit to its start vertex). In undirected, at most twice (once for each endpoint). So total number of times the loop runs, over all calls to recurse, is $O(m)$.
- Each loop run takes $O(1)$ time to find the next edge, check membership in reached, and set up a recursive call.
- Total over the whole algorithm: $n$ calls to recurse $\times O(1)$ work outside the loop $+ O(m)$ runs of the loop $\times O(1)$ work inside the loop $= O(n + m) = O(m)$. 

Issues with DFS in web spider application

- When exploring parts of the web graph spread across multiple web servers, it will hit single servers in hard bursts rather than spreading the load across the servers.

- Doesn’t deal well with implicit graphs that are not finite.

  Example: In the root folder of your web site, set up a symbolic link named “nest” that points back to the root folder itself. Now you have a site with infinitely many pages of the form “http://site/nest/nest/nest/nest/nest/…”

  Spider will get stuck in this trap and never return to the rest of the web.
Breadth-first search
Breadth-first search

Probably a better choice for web spiders:

- Also linear time, $O(m)$
- Only slightly more complicated to implement
- Explores vertices in order by number of steps from start, spreading load among multiple servers
- For nested web sites, will continue exploring rest of the web so everything else is eventually reached

Intuitive idea:

- Queue (first-in first-out) data structure of vertices to explore
- Explore vertices in queue order
- When we find an edge to an unreached vertex add it to the queue
def reachable(G,s):
    // arguments are an implicit graph G
    // and the starting vertex s

    reached = new set containing s
    queue = new queue containing s

    while queue is not empty:
        remove first item from queue; call it v
        for each edge in G from v to w:
            if w is not already in reached:
                add w to both reached and queue

    return reached
Breadth-first reachability example

```
queue: s; reached: s
remove s
add unreached neighbors a,b
queue: a,b; reached: s,a,b
remove a
add unreached neighbor d
queue: b,d
reached: s, a, b, d
remove b
add unreached neighbors c, e
queue: d, c, e
reached: s, a, b, d, c, e
remove d, c, e
(no more unreached neighbors)
```

Tree in bottom shows which \( v \) was the first to find each vertex \( w \), but it is also a tree in the graph, with shortest paths from \( s \) to other vertices

It is called the BFS tree

The algorithm can be modified to construct and return it
Analysis of breadth-first search

Even though it’s a loop rather than a recursion, it’s mostly the same as DFS.

Each vertex is only added to the queue when not in reached, and at the same time is added to reached, so added once per vertex \(\Rightarrow\) also removed once per vertex.

Each edge is looped over only when we remove one of its endpoints from the queue \(\Rightarrow\) once or twice per edge.

Constant work for each vertex or edge that we loop over \(\Rightarrow\) \(O(m)\) time.
Other exploration orders are also possible
The web graph is already in a graph representation, an implicit graph, even before we start exploring it.

So are the graphs (state spaces) of puzzles and planning problems.

Both depth-first and breadth-first search can be used to find all of reachable vertices of a (finite) graph in linear time.

Breadth-first may work better for the web graph (less bursty network traffic, better behavior when faced with infinite graphs).