Name: Key

UCInetID:
1. (12 points) Every undirected graph either has a graph coloring that uses only two colors, or an odd cycle, but not both. The graphs that have a 2-coloring are the bipartite graphs. Is the graph below bipartite? Find a 2-coloring or an odd cycle in it.

2. (12 points) Let $G$ be a graph with five vertices labeled $p, q, r, s, t$, in which every pair of vertices is adjacent except $q-s$.

(a) Find a planar drawing of $G$. Count how many vertices ($V$), edges ($E$), and faces ($F$) it has, and show that they obey Euler’s formula $V - E + F = 2$.

(b) Draw the dual graph of $G$ (note: not the complement graph!) Calculate its number of vertices, edges, and faces, and show that they again obey Euler’s formula.
3. (12 points) Every directed graph either has a topological ordering, or a directed cycle, but not both. The graphs that have a topological ordering are the directed acyclic graphs. Is the graph below a directed acyclic graph? Find a topological ordering or a directed cycle in it.

Yes.
there are many topological orders —
one possibility is

a b e f s r q p c (k j g c d)

4. (12 points) In what order would the Prim–Dijkstra–Jarník algorithm add edges to the minimum spanning tree of the following graph, if it selects k as its starting vertex? (List the weights of the spanning tree edges, in this order. Do not list weights that are not used in the spanning tree.)

41 22 31 21 42 23 32 24
5. (12 points) Every undirected graph either has an elimination ordering, or a cycle longer than three with no diagonals, but not both. The graphs that have an elimination ordering are the chordal graphs. Is the graph below chordal? Find an elimination ordering or a long diagonal-free cycle in it.

6. (12 points)

(a) As a function of the number $n$ of vertices, what is the largest possible number of vertices that can be in a maximum independent set in a graph that has a perfect matching?

\[ \frac{n}{2} \]

(b) As a function of the number $n$ of vertices, what is the smallest possible size of a maximum independent set in a bipartite graph?

\[ \left\lceil \frac{n}{2} \right\rceil \]
7. (12 points) Find a degeneracy ordering of the following graph. What is its degeneracy, and how many vertices are in its 2-core?

![Graph Image]

There are many degeneracy orders - one is: CGKRAHDJBEF

These 3 must be 1st; A or H next; E or F first, then the other; 4 in either order.

Degeneracy = 3
2-core has 8 vertices

8. (12 points) Suppose that you need to find shortest paths from a given starting vertex to a given destination in strongly connected directed graphs in which all edges have the same weight (weight = 1). Name the algorithm covered in this class that can solve this problem the fastest. What is its running time, using $O$-notation, as a function of $n$ (the number of vertices) and $m$ (the number of edges)?

BFS $O(m+n)$
9. (12 points) In a stable matching problem between people and positions, people A and B are looking to be matched to positions X and Y. There are two possible ways of matching the people to the positions. Construct a system of preferences for the people (for each person, which position they prefer) and for the positions (which person they prefer) such that both of these two matchings are stable.

-X: AB
- Y: BA

-A: YX
- B: XY

or

-X: BA
- Y: AB

-A: XY
- B: YX

10. (12 points) In a maximum flow problem from s to t in the flow graph drawn below, draw the residual graph after augmenting the flow (starting from a zero flow) along path s-a-d-c-b-t, sending as much flow along that path as possible.
Key
(problems 3 and 6)
1. (12 points) Every undirected graph either has a graph coloring that uses only two colors, or an odd cycle, but not both. The graphs that have a 2-coloring are the bipartite graphs. Is the graph below bipartite? Find a 2-coloring or an odd cycle in it.

![Graph](image)

2. (12 points) Let $G$ be a graph with five vertices labeled $p, q, r, s,$ and $t$, in which every pair of vertices is adjacent except $q-s$.

   (a) Find a planar drawing of $G$. Count how many vertices ($V$), edges ($E$), and faces ($F$) it has, and show that they obey Euler's formula $V - E + F = 2$.

   (b) Draw the dual graph of $G$ (note: not the complement graph!) Calculate its number of vertices, edges, and faces, and show that they again obey Euler's formula.
3. (12 points) Suppose you have access to a subroutine that tells you the strongly connected components of a graph (where each component is represented as a list of vertices, and the output of the subroutine is a list of lists). You do not have direct access to the graph itself; the only information you have about it is given by this subroutine. Describe in English (not pseudocode) how to use this information to test whether the graph is a directed acyclic graph.

It's a DAG if and only if every strongly connected component had exactly 1 vertex.

4. (12 points) In what order would the Prim–Dijkstra–Jarník algorithm add edges to the minimum spanning tree of the following graph, if it selects $k$ as its starting vertex? (List the weights of the spanning tree edges, in this order. Do not list weights that are not used in the spanning tree.)
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![Graph Image]

6. (12 points)

(a) As a function of the number $n$ of vertices, what is the largest possible number of vertices that can be in a maximum independent set in a graph that has a perfect matching? Describe a graph that has a perfect matching and whose maximum independent set has this size.

$$\frac{n}{2}$$

any bipartite graph with a perfect matching e.g. $K_{\frac{n}{2}, \frac{n}{2}}$

(b) As a function of the number $n$ of vertices, what is the smallest possible size of a maximum independent set in a bipartite graph? Describe a bipartite graph whose maximum independent set has this size.

$$\frac{\sqrt{n}}{2}$$

the complete balanced bipartite graph $K_{\frac{n}{2}, \frac{n}{2}}$
7. (12 points) Find a degeneracy ordering of the following graph. What is its degeneracy, and how many vertices are in its 2-core?

8. (12 points) Suppose that you need to find shortest paths from a given starting vertex to a given destination in strongly connected directed graphs in which all edges have the same weight (weight = 1). Name the algorithm covered in this class that can solve this problem the fastest. What is its running time, using $O$-notation, as a function of $n$ (the number of vertices) and $m$ (the number of edges)?
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