Integer priority queues
Main idea

$k$-ary heaps, Fibonacci heaps only use comparison of priorities

If priorities are small integers, we can use bitwise binary operations, array indexing, etc

Sometimes this can be faster
Integer priority queues versus sorting

Many non-comparison-based sorting algorithms are known

E.g. radix sort: Sort $n$ integers in range $[0, X]$ in time $O\left(\frac{n \log X}{\log n}\right)$

If we can do priority queue operations in time $T$, we can use them to sort in time $O(nT)$ (“heap sort”)

Reverse is true!

If we can sort in time $nT(n)$, we can do priority queue operations in time $T(n) + T(\log n) + T(\log \log n) + \cdots$

Thorup, “Equivalence between priority queues and sorting”, JACM 2007
Theoretically fast but completely impractical

Current best known (expected) time bound for integer sorting, when priorities are small enough to be represented by machine integers: $O(n \sqrt{\log \log n})$

Han & Thorup, “Integer sorting in $O(n \sqrt{\log \log n})$ expected time and linear space”, FOCS 2002

Corollary: Can do integer priority queue operations in $O(\sqrt{\log \log n})$ expected time per operation

We will look at simpler data structures whose time depends on the range of values, not just on $n$

Open: Can we do better or is there a non-constant lower bound?
Bucket queue

Priority queue version of bucket sort

When all priorities are small non-negative integers:

- Make an array of buckets, indexed by priority
- Each bucket: collection of elements with that priority
- Add element; put it into the bucket for its priority
- Change priority: move it from one bucket to another
- Find-minimum: scan array for first non-empty bucket

Time: $O($current minimum priority$)$ for find-min, $O(1)$ otherwise
Monotonic priorities and Dial’s algorithm

In Dijkstra’s algorithm, priorities never decrease

⇒ Can start scan for non-empty bucket at the last one we found, instead of starting at the beginning of the array

Total time for priority queue operations:
$O(\# \text{ operations} + \text{max priority})$

Total time for shortest paths in graphs with integer edge lengths:
linear in size of graph + $O(\text{max path length})$

Dial, “Algorithm 360: Shortest-path forest with topological ordering”,
CACM 1969
Flat trees
(also known as Van Emde Boas trees)
Think of priorities as $b$-bit binary numbers, for some $b$

Partition each priority $x$ into two $b/2$-bit numbers (the two digits in its base-$2^{b/2}$ representation)

- $x_{\text{low}} : x \& ((1 \ll b/2) - 1)$
- $x_{\text{high}} : x \gg (b/2)$

Build recursive priority queues for the low parts and the high parts

Each operation recurses into low or high but not both

$\Rightarrow$ time $O(\log b) = O(\log \log (\text{max priority}))$
Simplifying assumption: elements $\equiv$ priorities

If we have elements that are not the same thing as the priorities of the elements, then:

- Make a dictionary mapping each priority to a collection of elements with that priority (like bucket queue)
- Keep a priority queue of the set of priorities whose collections are non-empty
- To find an element of minimum priority: use the priority queue to find the priority, and the dictionary to find the element

The dictionary part is $O(1)$ expected time per operation (using a hash table) so we only need to worry about priority queues whose elements are integers and whose priorities are their values
The flat tree for a set $S$ of integers stores:

- The minimum element in $S$
- Recursive flat tree $H$ of $x_{\text{high}}$ for $x \in S \setminus \{\text{min}\}$
- Dictionary $D$ mapping each $x_{\text{high}}$ to a recursive flat tree of values $x_{\text{low}}$ for the subset of elements $x \in S \setminus \{\text{min}\}$ that have that value of $x_{\text{high}}$

To find the minimum priority:
Look at the value stored as minimum at root-level structure

To make one-element flat tree:
Set minimum to element, $H$ to None, $D$ to empty dictionary
Example

Suppose we are storing a flat tree for the nine 8-bit keys 00000001, 00000011, 00001010, 00001111, 10100101, 10101010, 11011110, 11011111, and 11101010. Then:

The minimum element is 00000001

The values $x_{\text{high}}$ for the remaining keys are 0000, 1010, 1101, and 1110. The recursive flat tree $H$ stores these four 4-bit keys.

The dictionary $D$ maps each of these four values $x_{\text{high}}$ to a recursive flat tree $D[x_{\text{high}}]$:

- $D[0000]$ stores the low 4-bit keys whose high 4 bits are 0000, not including the minimum element 00000001. These 4-bit keys are: 0011, 1010, and 1111.
- $D[1010]$ stores the two low 4-bit keys 0101 and 1010
- $D[1101]$ stores the two low 4-bit keys 1110 and 1111
- $D[1110]$ stores as its single element the low 4-bit key 1010.
To insert an element $x$:

1. If $x < \text{minimum}$, swap $x$ with minimum
2. If $x_{\text{high}}$ is in $D$:
   Recursively insert $x_{\text{low}}$ into $D[x_{\text{high}}]$
3. Else:
   Make one-element flat tree containing $x_{\text{low}}$
   Add the new tree to $D$ as $D[x_{\text{high}}]$
   Recursively insert $x_{\text{high}}$ into $H$
   (or make one-element flat tree for $x_{\text{high}}$ if $H$ was empty)
To find the second-smallest element

1. If $D$ is empty, raise an exception
2. Let $x_{\text{high}}$ be the minimum stored in $H$
3. Let $x_{\text{low}}$ be the minimum stored in $D[x_{\text{high}}]$
4. Combine $x_{\text{high}}$ and $x_{\text{low}}$ and return the result

Expected time $O(1)$

Key subroutine in element removal
To remove an element \( x \):

1. If \( x \) is the stored minimum:
   
   Set \( x \) and stored minimum to second-smallest value
   
   (still need to remove from recursive structure,
    just as if we were deleting it)

2. Compute \( x_{\text{high}} \) and \( x_{\text{low}} \)

3. If \( D[x_{\text{high}}] \) is a one-element flat tree (its recursive \( D \) is empty):
   
   Recursively delete \( x_{\text{high}} \) from \( H \)
   
   Remove \( x_{\text{high}} \) from \( D \)

4. Else:

   Recursively delete \( x_{\text{low}} \) from \( D[x_{\text{high}}] \)
Summary
Summary

- Priority queue operations, application in Dijkstra’s algorithm
- Limited set of operations in Python and how to work around it
- Binary heaps, $O(\log n)$ time per operation, $O(n)$ heapify
- $k$-ary heaps, slower delete-min and faster decrease-priority
- Optimal choice of $k$ for $k$-ary heaps in Dijkstra’s algorithm
- Fibonacci heaps, $O(\log n)$ delete-min, $O(1)$ decrease-priority
- Integer priority queues and their relation to integer sorting
- Bucket queues and Dial’s algorithm
- Flat trees, $O(\log \log (\text{max priority}))$ per operation