CS 261: Data Structures

Week 8: Navigating in trees

Lecture 8a: Succinct trees and level ancestors

David Eppstein
University of California, Irvine

Spring Quarter, 2023

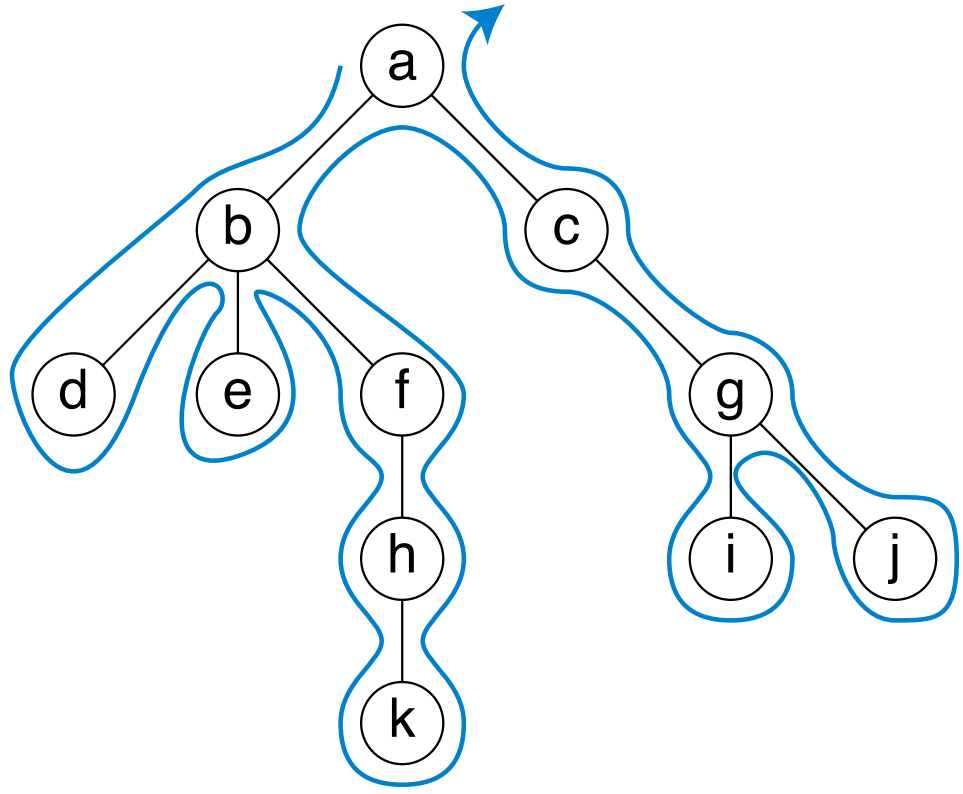
Euler tours in trees

Euler tours (intuition)

Draw a tree in the plane, from the root down

Trace your finger around the boundary of the drawing

What order does it reach each vertex (multiple times)?



a, b, d, b, e, b, f, h, k, h, f, b, a, c, g, i, g, j, g, c, a

Each edge traced twice $\Rightarrow$ # vertices in tour $= 2n - 1$

Euler tours (pseudocode)

To produce an Euler tour of a subtree rooted at x:

- Output x
- For each child y of x:
  - Recursively produce an Euler tour for the subtree rooted at y
  - Output x again

Like

preorder (root comes before children),
inorder (root comes between children), and
postorder (root comes after children),

all merged into one

Representing trees succinctly

Basic tree navigation operations

If we represent trees with an object for each node, we can handle:

- Find the root
- Check if a node is the root, or is a leaf
- Go to the parent of a node
- Go to the first child of a node
- Go to the next child of the same parent

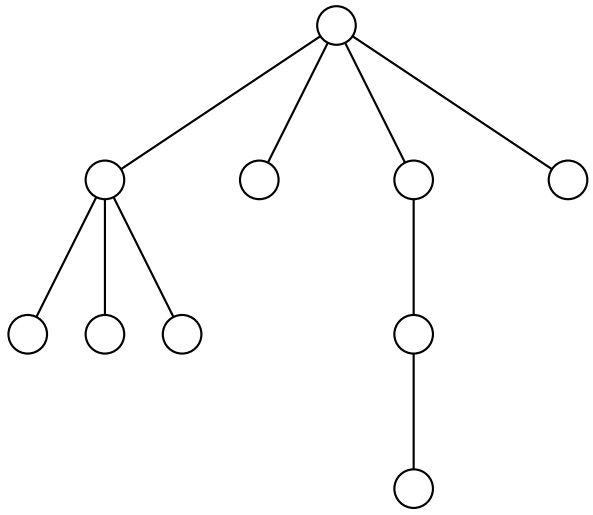
. . . by including a pointer to each of these things in each node

Space is $O(n)$ words of memory . . . but we can do better!

Arbitrary trees with ordered children

Write nested parentheses for each node,
surrounding string representing the subtree below it

Convert to binary: "(" = 0, ")" = 1



0 00101011 01 000111 01 1
((( ) ( ) ( )) ( ) ((( ))) ( ))

left child and its
three grandchildren

n-node tree $\Rightarrow$ $2n$-bit binary number

Open parenthesis: parent-to-child edge in Euler tour

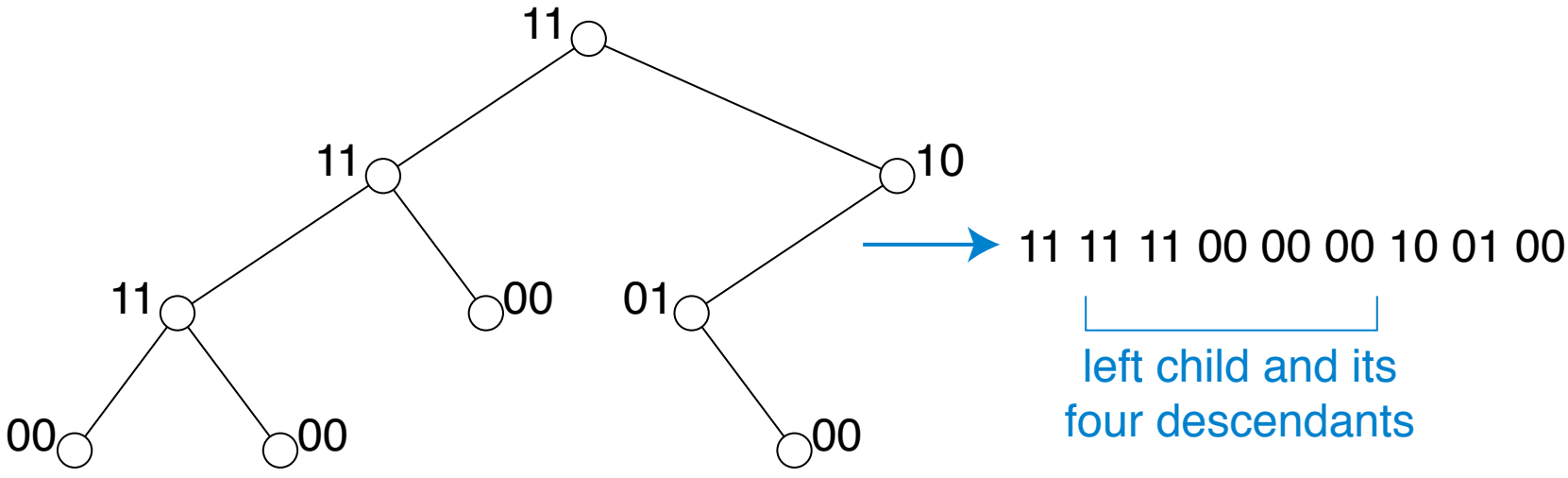
Close parenthesis: child-to-parent edge

Binary trees

Parentheses don't work: can't tell if single child is left or right

Traverse tree in preorder (root first, then children)

For each node, write down two bits:
Does it have a left child (0 for no, 1 for yes)?
Does it have a right child (0 for no, 1 for yes)?



n-node tree $\Rightarrow$ $2n$-bit binary number

Both codes are near-optimal

In each case, # trees is a Catalan number

$$C_n = \binom{2n}{n} \Big/ n+1 \approx \frac{4^n}{\sqrt{n^3\pi}}$$

(C_n for n-node binary trees, C_{n-1} for ordered trees)

So in order to have a different code for each tree, # bits must be

$$\geq \log_2 C_n = 2n - O(\log n)$$

Blocking

One long sequence of parentheses is concise but difficult to navigate

Break it up into blocks of $\frac{1}{2}\log n$ parens (represented as a sequence of $\frac{1}{2}\log n$ bits)

There are $\sqrt{n}$ different sequences of $\frac{1}{2}\log n$ bits $\Rightarrow$ we can build a table indexed by these sequences that lets us navigate within a block in constant time per step (tables take much less than n space to store)

There are $O(n/\log n)$ blocks $\Rightarrow$ we can use a less space-efficient structure to navigate from block to block and the small number of blocks will make it more efficient

We will see this same blocking strategy repeatedly all week!

Log-shaving principle

Using blocks can reduce memory requirement
from $S(n)$ to 2n bits + $S(n/\log n)$

Sometimes repeatable for $O(1)$ levels,
saving $\log n$ in space each time

The level-ancestor problem

The level-ancestor problem

Data: A tree

Query:

- Given a vertex v and a number k, find the ancestor of v that is k steps higher in the tree
- Equivalently: Given a vertex v and a number d find the ancestor whose depth (number of steps from root) is d

We could just follow parent links up the tree in time $O(k)$ but we want small query time even when k is large

Inefficient solution

Each node v stores $O(\log n)$ ancestors, the ones k steps higher for $k = 1, 2, 4, 8, \ldots 2^i, \ldots$

To find the ancestor k steps higher when k is not a power of two:

- Decompose k into a sum of powers of two (its binary representation)
- Use stored ancestor pointers to jump up by each power

Space $O(n \log n)$, query time $O(\log k)$

We can reduce space but first we need to speed up queries

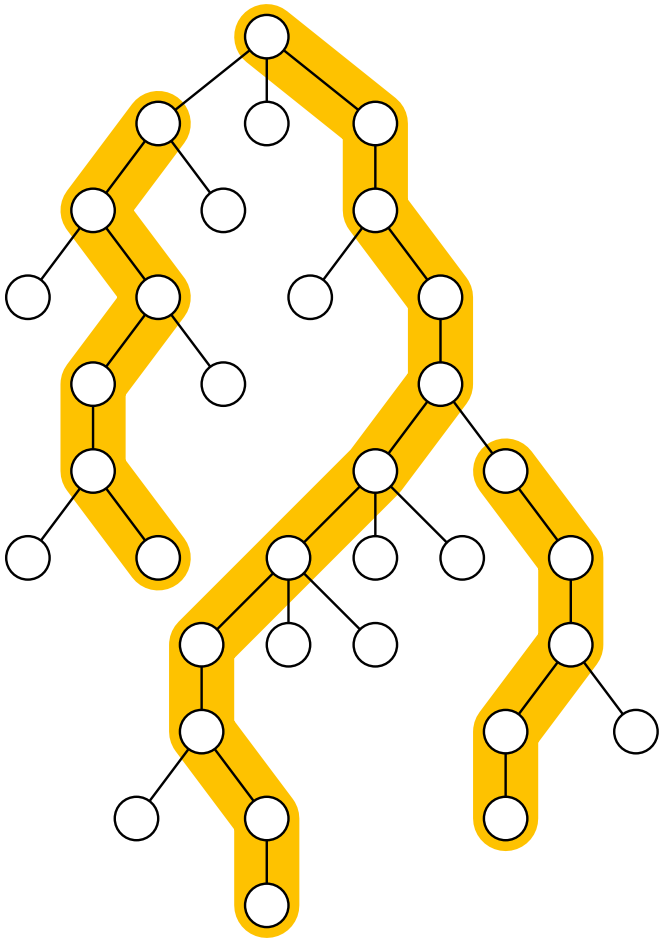
An easy special case

If the tree is a path, rooted at left end:

- Store an array A of the vertices in the path
- Each node records its position in the array
- If V is in $A[i]$, its k-step ancestor is in $A[i - k]$

Linear space, constant query time

Decomposition into long paths



Each non-leaf node selects one child, the one leading to the deepest leaf

The selected edges form a system of paths covering all non-leaf nodes and some of the leaves

Remaining leaves form one-vertex paths

But if we use the path solution for these paths, how do we find ancestors on different paths?

Extended paths

Instead of disjoint paths, extend each path to be twice as long (or all the way to the root if there is no ancestor twice as high)

Store each extended path in an array, and for each tree vertex store both which array it belongs to and its position in the array

Total length of all arrays $\leq 2n \Rightarrow$ linear space

Can answer some but not all queries: when v is h steps above a leaf, we can find ancestors h steps above v

Combined data structure

Store both power-of-2 ancestors and extended paths

To find the ancestor k steps higher from v:

- Make one power-of-2 jump, the biggest one that would still be below the ancestor

  Jump amount $= 2^{\lfloor \log_2 k \rfloor}$

- You are now high enough above a leaf to use extended paths

Constant query time but space is still $O(n \log n)$

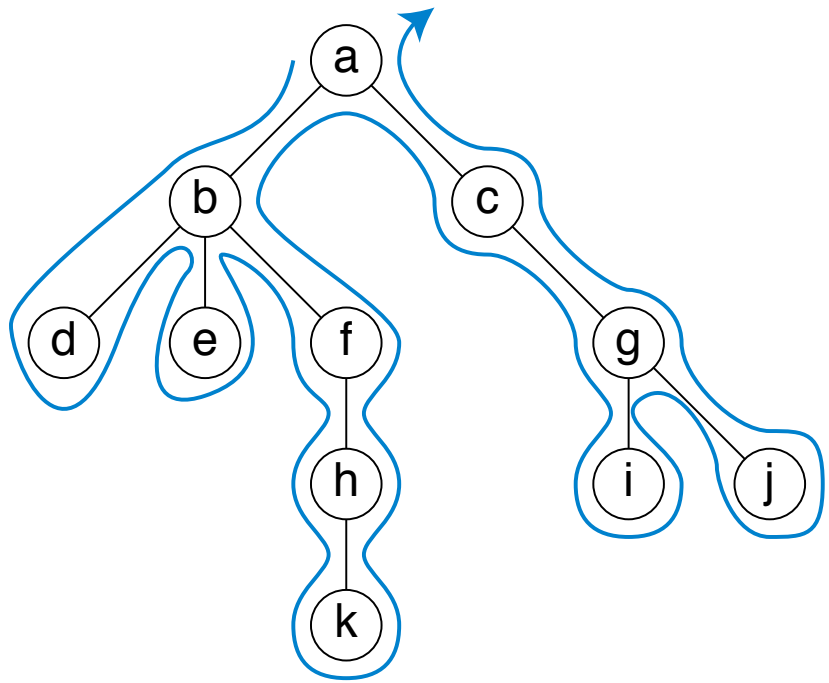
(The paths use linear space but the jump tables are too big)

[Bender and Farach-Colton 2004]

Level ancestors and Euler tours

To find the ancestor of v at height d:

- Look in the Euler tour of the tree, starting from the last copy of v in the tour
- The next vertex with height d is the ancestor



a, b, d, b, e, b, f, h, k, h, f, b, a, c, g, i, g, j, g, c, a

Log-shaving

Structure:

- Break Euler tour into blocks of length $b = \frac{1}{2} \log_2 n$
- Within block, depths differ by ± 1; label each block by its pattern of $\pm$ choices, a $(b-1)$-bit binary number
- Precompute tables for queries with answer in same block
- Store power-of-two tables only at the last vertex of each block

Query:

- Use table to find answer in v's block (if it is there)
- If not, vertex u at block end has same level-d ancestor as v
- Use the jump table stored at u to find an ancestor w high enough that we can use the extended paths for w

[Berkman and Vishkin 1994]

References and image credits

Michael A. Bender and Martin Farach-Colton. The level ancestor problem simplified. *Theoretical Computer Science*, 321(1):5–12, 2004. doi: 10.1016/j.tcs.2003.05.002.

Omer Berkman and Uzi Vishkin. Finding level-ancestors in trees. *Journal of Computer and System Sciences*, 48(2):214–230, 1994. doi: 10.1016/S0022-0000(05)80002-9.

liempdma. Cutting lumber with a swingblade sawmill. CC-BY-SA image, September 4 2018. URL `https://commons.wikimedia.org/wiki/File:Cutting_lumber_with_a_swingblade_sawmill.jpg`.