Which unit-side-length convex polygons can be formed by packing together unit squares and unit equilateral triangles? For instance one can pack six triangles around a common vertex to form a regular hexagon. It turns out that there is a pretty set of 11 solutions. We describe connections from this puzzle to the combinatorics of 3- and 4-dimensional polyhedra, using illustrations from the works of M. C. Escher and others.

(Joint work with Günter Ziegler)

1. Which convex polygons can be made from squares and triangles?
   2. Platonic solids
   3. The six regular 4-polytopes
   4. Mysteries of 4-polytopes
   5. Flatworms
   6. The puzzle solutions
   7. Polytopes and spheres
   8. Koebe's theorem
   9. Polarity
   10. The key construction
   11. E-polytopes
   12. Polars of truncated hypercubes?
   13. Hyperbolic space
   14. Models of hyperbolic space
   15. Size versus angle
   16. Right-Angled dodecahedra tile hyperbolic space
   17. Surprise!
   18. Dragon
Which strictly convex polygons can be made by gluing together unit squares and equilateral triangles?
The Six Regular 4-Polytopes

- **Simplex**, 5 vertices, 5 tetrahedral facets, analog of tetrahedron
- **Hypercube**, 16 vertices, 8 cubical facets, analog of cube
- **Cross polytope**, 8 vertices, 16 tetrahedral facets, analog of octahedron
- **24-cell**, 24 vertices, 24 octahedral facets, analog of rhombic dodecahedron
- **120-cell**, 600 vertices, 120 dodecahedral facets, analog of dodecahedron
- **600-cell**, 120 vertices, 600 tetrahedral facets, analog of icosahedron
Mysteries of four-dimensional polytopes...

What face counts are possible?

For three dimensions, $f_0 - f_1 + f_2 = 2$, $f_0 \leq f_2 - 4$, $f_2 \leq f_0 - 4$ describe all constraints on numbers of vertices, edges, faces. All counts are within a constant factor of each other.

For four dimensions, some similar constraints exist, e.g. $f_0 + f_2 = f_1 + f_3$ but we don’t have a complete set of constraints.

Is "fatness" $(f_1 + f_2)/(f_0 + f_3)$ bounded? Known $O((f_0 + f_3)^{1/3})$ [Edelsbrunner & Sharir, 1991]

How can we construct more examples like the 24-cell?

All 2-faces are triangles ("2-simplicial")

All edges touch three facets ("2-simple")

Only few 2-simple 2-simplicial examples were known: simplex, hypersimplex, 24-cell, Braden polytope
Octahedron and tetrahedron dihedrals add to 180!
So they pack together to fill space

M. C. Escher, Flatworms, lithograph, 1959
The Eleven Convex Square-Triangle Compounds
Polytopes and spheres

M. C. Escher, Order and Chaos, lithograph, 1950
Theorem [Koebe, 1936]:

Any planar graph can be represented by circles on a sphere, s.t. two vertices are adjacent iff the corresponding two circles touch.

Replacing circles by apexes of tangent cones forms polyhedron with all edges tangent to the sphere.
Polarity

Correspond points to lines in same direction from circle center
distance from center to line = 1/(distance to point)

Line-circle crossings equal point-circle horizon
Preserves point-line incidences! (a form of projective duality)

Similar dimension-reversing correspondence in any dimension
Converts polyhedron or polytope (containing center) into its dual
Preserves tangencies with unit sphere
Convex Hull of (P union polar), P edge-tangent
Edges cross at tangencies; hull facets are quadrilaterals

M. C. Escher, Crystal, mezzotint, 1947
Same Construction for Edge-Tangent 4-Polytopes?

Polar has 2-dimensional faces (not edges) tangent to sphere

Facets of hull are dipyramids over those 2-faces

All 2-faces of hull are triangles (2-simple)

Three facets per edge (2-simplicial) if and only if edge-tangent polytope is simplicial

This leads to all known 2-simple 2-simplicial polytopes

Simplex $\Rightarrow$ hypersimplex

Cross polytope $\Rightarrow$ 24-cell

600-cell $\Rightarrow$ new 720-vertex polytope, fatness=5

So are there other simplicial edge-tangent polytopes?
Polars of truncated hypercubes?

Formed by gluing simplexes onto tetrahedral facets of cross polytope

Always simplicial

Many different variations

If we warp the glued simplex to make it edge-tangent, is the result still convex?

Need a space where we can measure convexity independent of warping (projective transformations)

Answer: hyperbolic geometry!
Hyperbolic Space (Poincaré model)
Interior of unit sphere; lines and planes are spherical patches perpendicular to unit sphere

M. C. Escher, Circle Limit II, woodcut, 1959
Two models of Hyperbolic Space

Klein Model
Preserves straightness, convexity
Angles severely distorted

Poincaré Model
Preserves angles
Straightness, convexity distorted
Size versus angle in hyperbolic space

Smaller shapes have larger angles
Larger shapes have smaller angles

What are the angles in Escher's triangle-square tiling?

3 triangle + 3 square = 360
2 triangle + 1 square = 180
square < 90, triangle < 60

Another impossible figure!
Right-angled dodecahedra tile hyperbolic space

From Not Knot, Charlie Gunn, The Geometry Center, 1990
Surprise!

Edge-tangent cross polytopes have 90-degree hyperbolic dihedrals

Edge-tangent simplices have 60-degree hyperbolic dihedrals

So truncated cubes work! (new dihedrals are 150 degrees)

Other examples:

Six simplices around a triangle
(closely related to Soddy's hexlet of nine spheres in 3d)

Glue up to five cross polytopes around a central simplex
then close up nonconvexities by pairs of simplices

Even better, we get infinite families of simplicial edge-tangent polytopes, leading to infinitely many 2-simple 2-simplicial examples!

Glue $n$ cross polytopes end-to-end
forming $4n$ holes (180-degree dihedrals)
fill with $12n$ simplices, three per hole
M. C. Escher, Dragon, wood-engraving, 1952