# Horizon Theorems for Lines and Polygons

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**Abstract**

We give tight bounds on the complexity of the cells of a line arrangement that are cut by another line or by a convex polygon. These quantities are useful for the analysis of various geometric algorithms.

## 1. Introduction

A number of results in the analysis of algorithms depend on bounds on the complexity of *zones* in an arrangement; that is, given an arrangement of $n$ lines in the plane, and some figure in the same plane, we wish to know the sum of the numbers of sides of the cells in the arrangement that are cut by that figure. The basic result in this area is the so-called *horizon theorem*: the complexity of the cells supported by one side of any line in the arrangement is at most $5n$ [1, 2, 4]. This result can be used to prove an $O(n^2)$ bound on the time needed to construct the arrangement. The horizon theorem has also been used in some recent work on hidden surface removal and constructive solid geometry [6]; in these cases the figure cutting the arrangement is a convex $k$-gon, and we wish to know the complexity of the cells touching the $k$-gon on the inside. For fixed $k$ the previous bound shows that this is $O(n)$.

We give a number of results on the complexity of zones in an arrangement of lines:

1. The maximum number of sides in all cells supported by a single line is at most $9.5n+O(1)$, improving a previous bound of $10n$. We give an example to show that the new bound is tight up to $O(1)$. (We have recently learned that Edelsbrunner et al. have proved the same bounds with different techniques [3].)

2. We give examples showing that this bound does not generalize to other related configurations: the complexity of cells supported by two sides of two parallel lines, either between the two lines or on the outsides of the lines, can be at least $10n + O(1)$. By previously known results, this is also an upper bound.

3. The maximum complexity of the cells touching the inside of a triangle is at most $10.5n + O(1)$ and at least $10n + O(1)$.

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4. The complexity of the cells touching the inside of a convex $k$-gon, is at most $11n+(3/2)k^2$.

5. The complexity of the cells touching the inside of a convex $k$-gon is $O(n\alpha(n,k))$, assuming $k$ is $O(n)$. This gives a tighter bound than (4) for $k$ larger than about $\sqrt{n}$ and improves previous bounds of $O(n\alpha(n))$ [5] and $O(nk)$ [6]. Here $\alpha(n)$ and $\alpha(n,k)$ are one-variable and two-variable inverse Ackermann functions, respectively.

These results also hold for *pseudoline arrangements*. A pseudoline arrangement is a collection of curves, in which each pair of curves intersects at most once (at a crossing, rather than at a tangency). A convex $k$-gon cutting a pseudoline arrangement is assumed to cut each curve at most twice. In the final section of the paper, we give a sixth result. We adapt a construction that gives $n$ line segments with lower envelope complexity $\Omega(n\alpha(n))$ [10, 8] to show that the complexity of an arrangement of pseudolines cut by a convex $k$-gon may be as large as $\Omega(n\alpha(n,k))$. This construction proves that the upper bound in (5) is tight for pseudolines up to a constant factor. It is unknown whether the bound in (5) is tight for straight lines.

## 2. Tight Bounds for Both Sides of a Line

Let $\mathcal{A}$ be an arrangement of lines in the plane. Following Edelsbrunner [2], we define a *1-border* (respectively, *0-border*) of $\mathcal{A}$ to be a side (vertex) of a polygonal cell of $\mathcal{A}$. A 1-border can be thought of as a pair, consisting of a line segment and a cell. The *zone* of a line of $\mathcal{A}$ is the set of 0- and 1-borders that bound cells supported by that line. The complexity of a zone is its cardinality.

The number of 0-borders in the zone of a line or a convex polygon is exactly the number of 1-borders minus one for each unbounded cell. Since we are interested in the maximum possible complexity of a zone, and since there are constructions for zones achieving the maximum 1-border complexity that have only 4 unbounded cells, we can treat the 0-border complexity as essentially equal to the 1-border complexity. Therefore, we consider only 1-borders in all our complexity bounds.

We further assume that the $n+1$ lines of $\mathcal{A}$ are in general position. We refer to the $(n+1)$-*st* line $h_0$ whose zone is under consideration as the *horizon* line, and assume that it is horizontal. For a line $a$ of $\mathcal{A}$, $a \neq h_0$, we distinguish the two *sides* of $a$, left and right. A 1-border contained in $a$ belongs to one side or the other.

First we review an argument due to Edelsbrunner et al. [2, 4] that gives an upper bound of $5n-1$ on the number of 1-borders that lie on one side of horizon $h_0$ (that, say, lie in the closed half-plane above $h_0$). We conceptually sweep a horizontal line $h$ vertically away from $h_0$. During the sweep, each side of each line of $\mathcal{A}$, other than $h_0$, is in one of three states: *alive*, *sleeping*, or *dead*. Each side starts in the alive state, and transition rules determine the states resulting as $h$ passes through line intersections. Intuitively, a side is alive if its current intersection with sweep line $h$ is visible to horizon $h_0$; a side is sleeping if its intersection is currently invisible, but may become visible again as $h$ continues; and a side is dead if it is currently invisible and will remain invisible for the rest of the sweep.

Figure 1 illustrates the six transition rules as $h$ passes through the intersection of two lines. We show the states of only the left sides of the lines, since left sides and right sides interact independently and symmetrically, that is, the new state of $a$'s left side depends only on its old state and the old state of $b$'s left side.

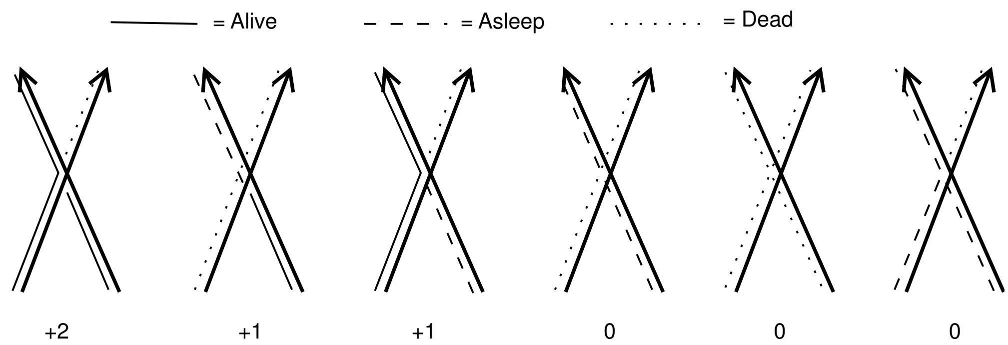


Figure 1. Transition rules for left sides of lines.

As the sweep proceeds we count the number of 1-borders in $h_0$'s zone that are not contained in $h_0$; a 1-border is added to the count when its upper endpoint is passed. (1-borders that extend to infinity are thrown in after the sweep has passed all intersections.) The numbers below each transition rule show the net gain in the count of left-side 1-borders.

Valuing an alive side at 2, a sleeping side at 1, and a dead side at 0, we see that the net gain in the count is never more than the loss in value. Since there are $n$ left sides, all initially alive, the total number of left-side 1-borders not contained in $h_0$ is at most $2n$. Adding $2n$ for right sides and $n+1$ for 1-borders contained in $h_0$, and then subtracting 2 since at least 2 sides remain alive forever (the right side with smallest positive slope and the left side with the smallest negative slope) gives $5n-1$. It is not hard to create an example that shows that this value is tight for the number of 1-borders above the horizon.

By simply doubling this value a bound of $10n-2$ can be obtained for the total number of 1-borders, above and below, in the zone of $h_0$ [1, 2, 4]. We now prove a tighter bound. Our strategy is to show that many sleeping sides never return to the alive state and hence leave unused value in the accounting scheme.

We must simultaneously consider both the above and below halfplanes. From now on, the word *side* means a side of a ray with vertex on $h_0$. Each line of $\mathcal{A}$ other than $h_0$ has upper right, upper left, lower right, and lower left sides. We write $a \times b$ for the intersection of lines $a$ and $b$, and we say $a \times b$ is *above* $c \times d$ if $a \times b$ has the larger $y$-coordinate.

**Definition 1.** A line $a$ of $\mathcal{A}$, $a \neq h_0$, is *full* if all four of its sides make a transition from alive to dead.

A line that is not full must have at least one side that either makes a transition from sleeping to dead or goes off to infinity sleeping or alive. Thus a line that is not full has unused value at least 1. We say that a side *does not wake up* if it makes a transition from sleeping to dead or goes off to infinity sleeping. We say that line $a$ *kills* a given side of line $b$ if at $a \times b$, that side of $b$ makes a transition from alive or sleeping to dead.

Call a region $w$ of the plane a *dead wedge* if $w$ is the intersection of two closed halfplanes bounded by lines of $\mathcal{A}$ and $w$ does not intersect $h_0$. Assume that $w$ lies above $h_0$, and name the two rays that bound $w$ right and left in the obvious way. The following lemma is immediate and applies analogously to the other 3 types of sides. See Figure 2.

**Lemma 1.** *Assume that at some point in the sweep, the upper right side of line $a$ lies within dead wedge $w$ and is sleeping. Then either $a$ intersects the right ray of $w$ or $a$'s upper right*

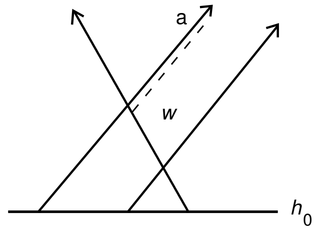


Figure 2. Traversing a dead wedge.

*side does not wake up.* ■

We now look at a full line $f$ in detail. The four sides of $f$ each make an alive-to-dead transition. Define lines $a$, $b$, $c$, and $d$ to be the lines which kill, respectively, the lower right, upper left, upper right, and lower left sides of $f$. Then $a$ and $b$ must form larger angles than $f$ with horizon $h_0$ (where the angle is counterclockwise between $h_0$ and the other line), while $c$ and $d$ form smaller angles. Assume further that $b \times f$ is above $c \times f$. See Figure 3.

**Lemma 2.** *Under the assumptions above:*
*(a) Intersection $a \times f$ is below $d \times f$.*
*(b) Intersection $a \times c$ lies below horizon $h_0$.*
*(c) Intersection $b \times d$ lies above horizon $h_0$.*

**Proof:** Assume the opposite of (a). Then since the lower left side of $f$ must be alive just above $f \times d$, intersection $a \times c$ must lie above $h_0$. But then the upper left side of $f$ cannot be alive just below $f \times b$.

Observation (b) follows from the fact that $f$'s upper left side must be alive just below $f \times b$. Similarly, observation (c) follows from the fact that $f$'s lower right must be alive just above $f \times a$. ■

Thus Figure 3 gives the only possible arrangement of lines $a$, $b$, $c$, $d$, $f$, and $h_0$, assuming that $b \times f$ is above $c \times f$. The opposite assumption, that is, $c$ above $b$ gives the mirror image of Figure 3.

**Lemma 3.** *There is a line $c^*$ satisfying the following: (1) $c^*$ intersects $f$ below $f \times b$ and at or above $f \times c$, (2) $c^*$ intersects $b$ at or below $b \times c$, (3) the upper right side of $c^*$ is alive just below $c^* \times b$ and is sleeping just above $c^* \times b$, and (4) $c^*$ crosses $h_0$ to the right of $h_0 \times a$ and at or to the left of $h_0 \times c$.*

**Proof:** Line $c$ is just such a line unless its upper right side is killed somewhere between its intersections with $f$ and $b$. The line that kills the upper right side of $c$ is suitable unless this line is itself killed. Following this chain of killers leads to a suitable $c^*$. Notice that (4) above holds for every line along the chain, since $f$'s upper left side is alive just below $f \times b$. ■

**Lemma 4.** *There is a line $d^*$ satisfying the following: (1) $d^*$ intersects $f$ above $f \times a$ and at or below $f \times d$, (2) $d^*$ intersects $a$ at or above $a \times d$, (3) the lower left side of $d^*$ is alive (sleeping) just above (below) its intersection with $d^* \times a$, and (4) $d^*$ crosses $h_0$ to the left of $h_0 \times b$ and at or to the right of $h_0 \times d$.*

**Proof:** Symmetric to the proof of Lemma 3. ■

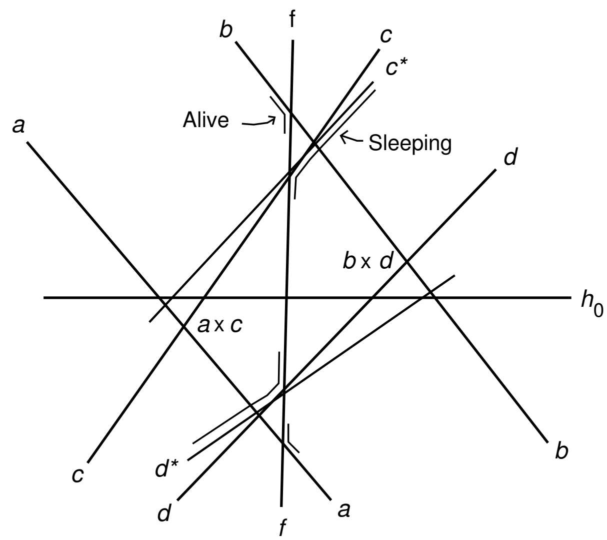


Figure 3. Line $f$ is full.

**Lemma 5.** *Assume $c^*$ satisfies (1)-(4) of Lemma 3 and $d^*$ satisfies (1)-(4) of Lemma 4. Then either the upper right side of $c^*$ or the lower left side of $d^*$ does not wake up.*

**Proof:** Lines $c^*$ and $d^*$ intersect either above or below $h_0$. If they intersect below, then $c^*$ does not intersect the right ray of the dead wedge defined by $b$ and $d^*$ above the horizon. If they intersect above, then $d^*$ does not intersect the left side of the dead wedge defined by $a$ and $c^*$ below the horizon. Lemma 1 now implies the result. ■

The lemmas above show how to associate a full line $f$ with two sleeping sides, i.e., the upper right of $c^*$ and the lower left of $d^*$, such that one of them does not wake up. For the mirror image case, that is, when $c \times f$ is above $b \times f$, the associated sides are an upper left and a lower right.

Define a mapping $ur$ from full lines to upper right sides that maps a full line $f$ to its associated side of $c^*$. Notice that $c^*$ is well-defined by the procedure given in the proof of Lemma 3. Similarly define a mapping $ll$ from full lines to lower left sides that maps a full line $f$ to its associated side of $d^*$. For the mirror image case, there are mappings $ul$ and $lr$.

Now assume that $f$ and $g$ are distinct full lines such that $ur(f) = ur(g)$ and these sides are contained in line $c^*$. Rename $f$ and $g$ if necessary so that $c^* \times f$ is above $c^* \times g$. See Figure 4.

**Lemma 6.** *If $ur(f) = ur(g)$ and $c^* \times f$ is above $c^* \times g$, then $ll(g) \neq ll(f)$ and $ll(g)$ does not wake up.*

**Proof:** Let $a_f$ and $b_f$ denote the lines that kill $f$'s lower right and upper left sides, and $d^*_f$ denote the line containing $ll(f)$. Analogously name $a_g$, $b_g$, and $d^*_g$. In Figure 4, the "diamond" formed by $g$'s four associated lines is shaded.

Notice that $g \times f$ must be above $c^* \times f$ and below $b_f \times f$, because $f$ is alive just below $f \times b_f$. Next, $b_g \times g$ must be above $c^* \times g$ and at or below $f \times g$. In Figure 4, $b_g = f$. Then the fact that $b_g \times d^*_g$ is above $h_0$ (as in Lemma 2(c)) implies that $d^*_g \times h_0$ is to the left of $f \times h_0$. Hence $d^*_g \neq d^*_f$. Now since $c^*$ is alive just below $b_f \times c^*$, $d^*_g$ must cross $c^*$ above $h_0$ and below

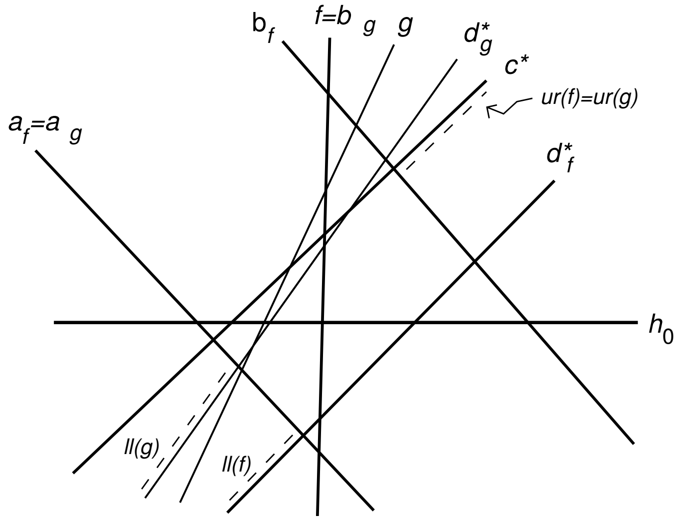


Figure 4. Full lines $f$ and $g$ are such that $ur(f) = ur(g)$.

$b_f \times c^*$. Then the lower left side of $d_g^*$ does not wake up below $d_g^* \times a_g$, since to do so it would have to cross $c^*$ below the horizon $h_0$. ■

**Lemma 7.** *Statements symmetric to Lemma 6 hold for mappings $ll$, $ul$, and $lr$.* ■

Roughly speaking, Lemma 6 shows that even though $ur$ is not one-to-one, if two full lines $f$ and $g$ map to the same upper right side, there are still two sides that do not wake up to "blame"—$ll(g)$ and one of $ur(f)$ and $ll(f)$.

**Lemma 8.** *The number of full lines is no greater than the number of sides that do not wake up.*

**Proof:** Consider the full lines whose upper left sides are killed above their upper right sides, such as $f$ in Figure 3. Number such full lines $f_1, f_2, \ldots$ clockwise, that is by decreasing angle with $h_0$. Either $ur(f_1)$ and $ll(f_1)$ are unique to $f_1$ or there are full lines that share $ur(f_1)$ as their $ur$ images and/or lines that share $ll(f_1)$ as their $ll$ images. If unique, then Lemma 5 implies that one of these sides does not wake up, so assume that at least one of $ur(f_1)$ and $ll(f_1)$ is not unique to $f_1$. Notice that a line that shares its $ur$ (respectively $ll$) image with $f_1$ must intersect $f_1$ above (below) $h_0$, so these two sets of lines are disjoint. By Lemma 6, a line $f_i$ that shares its $ur$ image with $f_1$ has an $ll$ image that does not wake up.

Image $ll(f_i)$ may itself be shared with another full line $f_j$. We assert that $j$ must be larger than $i$. To see this assertion, consider Figure 4 again. Let $f$ in the figure be $f_1$ and $g$ be $f_i$. Recall that the arrangement of $f = f_1$, $g = f_i$, $h_0$, and $c^*$ is the only possible arrangement such that $ur(f_1) = ur(f_i)$. Now assume $f_j$ is such that $ll(f_j) = ll(f_i)$, and $f_j$ forms a larger angle with $h_0$ than $f_i$, i.e., $j < i$. Since $ll(f_j) = ll(f_i)$, the lower left side of $f_j$ must be crossed by $d_g^*$ (the line containing $ll(f_j)$) below $h_0$. And $h_0 \times f_j$ must be to the right of $h_0 \times c^*$, or else $c^*$ would kill $f_j$'s lower left side. But now the upper left side of $f_i = g$ cannot be alive above $c^*$ as it lies in the dead wedge formed by $f_j$ and $c^*$. This contradicts the fact that $f_i$ is full.

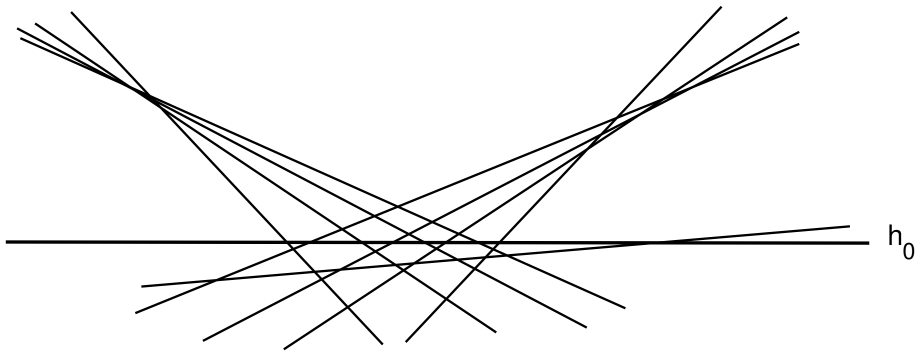


Figure 5. A $9.5n + O(1)$ example for two sides of one line.

Starting with $f_1$, we can define a rooted tree of full lines, in which a parent full line shares one of its images under $ur$ and $ll$ with each of its children. This procedure defines a tree because, by the assertion above and a symmetric counterpart, each full line adds children of larger index than its own index. Lemmas 6 and 7 guarantee that each full line added to the tree defines a new side that does not wake up. After a tree terminates we start another tree with the first $f_i$ that has not yet participated. A symmetric argument matches mirror image full lines with $ul$ and $lr$ sides that do not wake up. ■

**Theorem 1.** *The maximum number of 1-borders in $h_0$'s zone is at most $\lfloor 9.5n \rfloor - 1$.*

**Proof:** Let the number of full lines be $F$, the number of sides that do not wake up be $W$, and the maximum number of 1-borders in $h_0$'s zone be $M$. There are $2n + 2$ 1-borders contained in $h_0$, and the total initial value in the two sweeps is $8n$. There are 2 lines—the "most horizontal" lines—each of which contains 2 sides that go off to infinity alive.

Recall that each line that is not full has unused value at least one, and the two most-horizontal lines—not full—have unused value 2 each. Hence, $M \leq 9n + F$. A side that does not wake up has unused value one, so we also have the inequality $M \leq 10n - 2 - W$. By Lemma 8, $F \leq W$. The minimum of $9n + F$ and $10n - 2 - W$ is hence no more than $\lfloor 9.5n \rfloor - 1$. ■

Figure 5 gives an example with $\lfloor 9.5n \rfloor - 3$ 1-borders. Figures 6 and 7 show that Theorem 1 cannot be generalized to the complexity of cells on opposite sides of two parallel lines—in both possible cases the complexity can be as much as $10n + O(1)$. In these figures, a few straight lines are shown curved in order to fit them on the page. These examples generalize in the obvious way to other sufficiently large $n$.

## 3. Bounds for a Triangle

In the remainder of this paper, we consider the case of an arrangement of lines cut by a convex $k$-gon. There are two kinds of cells in the zone of this configuration: those outside the $k$-gon, and those inside the $k$-gon. Note that the $10n + O(1)$ upper bound for the zone of a line also holds, with minor modifications, for the complexity of the outside cells in the $k$-gon zone. We shall focus only on those cells inside the $k$-gon.

As with the previous bounds, it makes sense to allow the lines of the arrangement and the sides of the $k$-gon to be generalized to pseudolines. The requirements are: each pair of

Figure 6. A $10n + O(1)$ example for the insides of two parallel lines.

Figure 7. A $10n + O(1)$ example for the outsides of two parallel lines.

pseudolines intersects at most once, and they must cross at intersections; and each pseudoline of the arrangement crosses exactly two $k$-gon sides, crossing each side only once. The second requirement replaces the condition in the straight line case that the $k$-gon be convex.

In this section we give an upper bound of $10.5n + O(1)$ on the zone complexity of the inside of a triangle. It is easy to adapt Figure 6 to give a lower bound of $10n + O(1)$ in which no lines intersect one side of the triangle.

Let the sides of the triangle be denoted $s_1$, $s_2$, and $s_3$ clockwise. Let the lines cutting side $s_i$ be $N_i$, with $|N_1| + |N_2| + |N_3| = 2n$. Consider any single side, say $s_1$. The 1-borders supported by $s_1$ are of two types: those contained in lines of $N_1$ and those contained in lines of $N_2 \cap N_3$. Let $c_1$ denote the lower envelope of $N_2 \cap N_3$ inside the triangle; $c_1$ is a convex polygonal chain with $|c_1|$ segments and endpoints on $s_2$ and $s_3$. We can similarly define chains $c_2$ and $c_3$.

Convex chain $c_i$ is a pseudoline with respect to the lines in $N_i$, that is, each line of $N_i$ intersects $c_i$ at most once, and at that intersection crosses $c_i$. Consider sweeping a parallel line away from $s_i$ as in the argument of Edelsbrunner et al. reviewed in the last section. We may assume that $s_i$ is horizontal, so that left and right are canonically defined.

**Lemma 9.** *The complexity of cells supported by $s_i$ is at most $5|N_i| + |c_i|$.*

**Proof:** First note that if $|c_i| = 0$, the lemma is true by the $5n - 1$ bound, so assume $|c_i| \geq 1$. The complexity of cells supported by $s_i$ is the complexity of the arrangement formed by $N_i$ and $c_i$, considered as a pseudoline, plus the number of corners $|c_i| - 1$. In the sweep away from $s_i$, pseudoline $c_i$ adds only 2 to the complexity instead of the usual 5, because only one of its sides is visible and it does not cut $s_i$. Thus we have a bound of $5|N_i| + 1$. Considering $c_i$ as a polygonal chain rather than a pseudoline adds $|c_i| - 1$. ■

Figure 8. A line that kills a side of a full line cannot be full.

**Definition 2.** A line $f$ of $N_i \cap N_j$ is *full* if (1) it contributes a segment to $c_k$, $k \neq i, j$; (2) it makes 4 transitions (left and right sides on each of $s_i$ and $s_j$) from alive to dead; and (3) there is a point $p$ on the segment of $f$ on $c_k$ such that $p$ is visible from $s_k$ but not visible from $s_i$ or $s_j$, that is, $p$ lies between the two alive-to-dead transitions of the side of $f$ facing $s_k$.

A 1-border can be counted either in one of the three sweeps, or as part of one of the convex chains. Requirement (3) above ensures that a full line contributes an "extra" segment to a convex chain, that is, one that is not already counted in one of the sweeps. A *nonfull* line either has unused value in one of the two sweeps, as in the last section, or it does not contribute an extra visible corner to a convex chain $c_i$.

**Lemma 10.** *Let $f$ be a full line that contributes to $c_2$. Let $a$ be the line that kills $f$'s left side in the sweep from $s_1$. Then $a$ is not full.*

**Proof:** Assume $a$ is full. Line $a$ cannot contribute to convex chain $c_2$, as this would contradict the requirement that some point of $f$ be visible from $s_2$ above $f \times a$. Here "above" means further along in the sweep from $s_1$. So assume $a$ contributes to $c_3$, as shown in Figure 8. First assume $a$'s point $p_a$ visible from $s_3$ (guaranteed by (3) above) lies below $f \times a$. Now the line that kills $a$'s left side below $p_a$ must also kill $f$'s left side below $f \times a$, a contradiction. So assume point $p_a$ lies above $f \times a$. Consider the line $b$ that kills $a$'s left side in the sweep from $s_2$. For $p_a$ to be visible from $s_3$, $b \times s_3$ must lie above the endpoint of $c_2$ on $s_3$. This arrangement contradicts the fact that some point of $f$ above $f \times a$ is visible from $s_2$. ■

**Lemma 11.** *Let $f$ be a full line that contributes to $c_2$, and let $a$ be the line that kills the right side of $f$ in the sweep from $s_1$. Then $a$ is not full.*

**Proof:** Assume $a$ is full. Line $a$ must intersect $s_3$, so $a$ must contribute to $c_2$. The point $p_a$ cannot lie above $a \times f$, so it must lie below $a \times f$ and above the intersection at which $a$'s right side dies. But then the line that kills the right side of $a$ must kill the right side of $f$ below $f \times a$. ■

**Lemma 12.** *The number of full lines in $N_i$ is at most $|N_i|/2$.*

**Proof:** Consider the set of full lines $F_i$ in $N_i$. By Lemmas 10 and 11 and symmetric counterparts, a line in $F_i$ cannot kill a side of another line in $F_i$. Now consider only the initial segments of lines in $F_i$ up to the lowest (in the sweep from $s_i$) intersection at which both sides are dead. There are no intersections among these line segments.

We now assert that it takes $|F_i|$ lines to kill all of these segments. To each initial segment $f$, assign the line of $N_i$ that kills a side of $f$ below $f$'s upper endpoint. Assume line $a$ is assigned twice: say $a$ kills in sweep order the right sides of segments $f$ and $g$. Then we have a contradiction to the assumption that the left side of $g$ is alive above $g \times a$. The case that $a$ kills two left sides is symmetric.

Finally, assume $a$ kills in sweep order the left side of $f$ and the right side of $g$. Then the line that kills the right side of $f$ at its upper endpoint cannot escape from the triangle formed by $a$, $g$, and $s_i$ without killing the right side of $g$ below $g \times a$, a contradiction. The case of a right side followed by a left side is symmetric. Thus no line that kills a side of $F_i$ is assigned twice, and we have $|F_i| \leq |N_i|/2$. ■

**Theorem 2.** *The complexity of the cells on the inside of a triangle, cut by $n$ lines, is at most $10.5n + O(1)$.*

**Proof:** Lemma 12 implies that the sum over $i$ of the number of full lines in $N_i$ is at most $n$. Since each full line cuts two sides, this means that the total number of full lines is at most $n/2$. Each line starts with value 9, that is, 2 for each live side in each of the sweeps from the sides of the triangle it cuts, and 1 more for the possibility of contributing a corner to a convex chain $c_i$. Each nonfull line has unused value at least 1. Thus the average used value is at most 8.5 and the total—including the $2n$ 1-borders contained in the triangle itself—is at most $10.5n + O(1)$. ■

**Open Problem 1.** *Reduce the gap between the lower bound of $10n + O(1)$ and the upper bound of $10.5n + O(1)$ on the complexity of the cells along the inside of a triangle.*

## 4. Polygons with Fixed Number of Sides

In this section and the next we give bounds on the complexity of the zone of a $k$-gon in an arrangement of pseudolines.

Define $V(n, k)$ to be the maximum possible number of pairs $(\ell, s)$, where $\ell$ is a line in the arrangement and $s$ is a side of the $k$-gon, where $\ell$ does not intersect $s$, but where $\ell$ forms the side of a cell supported by $s$. There may be many such cells, but still $(\ell, s)$ is counted only once. If $\ell$ forms the side of a cell supported by multiple $k$-gon sides, we only include the pair $(\ell, s)$ where $s$ is the most clockwise of the sides supporting the cell. (However, a pair $(\ell, s')$, where $s'$ is not most-clockwise, is included if $\ell$ is visible to $s'$ from another cell.) The function $V(n, k)$ is significant for the $k$-gon zone complexity because of the following fact.

**Lemma 13.** *The complexity of the cells on the inside of a $k$-gon, cut by $n \geq 1$ lines, is at most $10n + k + V(n, k)$.*

**Proof:** Let the sides of the $k$-gon be denoted $s_1$, $s_2$, …, $s_k$. Let the lines crossing side $s_i$ be denoted $N_i$. Let the lines visible to side $s_i$ but not crossing it be denoted $V_i$.

If side $s_i$ is not crossed by any line, it adds at most one to the total complexity of the figure; there are at most $k$ such sides.

First assume $V_i$ is empty. Then the cells supported by $s_i$ are exactly those in the arrangement of only the lines in $N_i$, and the complexity of these cells is at most $5|N_i| - 1 \leq 5|N_i| + |V_i|$.

Otherwise, the "lower envelope" of the lines in $V_i$ (i.e., assuming $s_i$ is horizontal and below all lines of $V_i$) forms a convex chain, that can be treated as a pseudoline with respect to the lines in $N_i$. The complexity of all cells supported by $s_i$ is then the complexity of the arrangement formed by $N_i$ together with this extra pseudoline, plus the complexity of the corners of the pseudoline. As in the last section, the arrangement complexity is at most $5|N_i| + 1$. The number of corners on the pseudoline is $|V_i| - 1$. Therefore in this case also the complexity of the cells supported by $s_i$ is bounded by $5|N_i| + |V_i|$.

The complexity of the whole arrangement (counting cells supported by exactly two sides twice), is then at most

$$k + \sum_{i=1}^{k}(5|N_i| + |V_i|) = k + 5\sum_{i=1}^{k}|N_i| + \sum_{i=1}^{k}|V_i| = 10n + k + V(n,k).$$

■

We now give a simple combinatorial lemma that we use to prove a bound on $V(n,k)$; this will then be used as a base case in our more complicated final bound.

**Lemma 14.** *Let $S_1$, $S_2$, ..., $S_m$ be a family of subsets of a set $S$ with $|S| = k$, with the property that, for some constant $a$ and all $i \neq j$, $|S_i \cap S_j| \leq a$. Then $\sum_{i=1}^{m}|S_i| \leq am + \binom{k}{a+1}$.*

**Proof:** Let $b_i = |S_i| - a$. Then each set $S_i$ has at least $b_i$ $(a+1)$-tuples, that by assumption must be distinct from the $(a+1)$-tuples in all the other $S_j$. So $\sum_{i=1}^{m} b_i \leq \binom{k}{a+1}$. Therefore $\sum_{i=1}^{m}|S_i| \leq am + \sum_{i=1}^{m} b_i \leq am + \binom{k}{a+1}$. ■

**Lemma 15.** $V(n,k) \leq 2n + k^2/2$.

**Proof:** Let $m = 2n$ in Lemma 14, and let the sets $S_i$ be the $k$-gon sides visible to each side of each line in the arrangement. As above, only the first side in a multi-side cell is "visible". Then, for two sets $S_i$ and $S_j$ that correspond to sides of distinct lines, the zone cells in the region inside both corresponding halfplanes can be ordered linearly around the $k$-gon. The only possible $k$-gon side shared by both sets is the one supporting the cell first visible to $S_j$ in this linear order; the view from a $k$-gon side later in the order is blocked by the view within this cell. Thus Lemma 14 applies with $a = 1$. ■

**Lemma 16.** $V(n,k) \leq n + (3/2)k^2 - k$.

**Proof:** Divide the lines into classes $C_{i,j}$, where line $\ell$ belongs to $C_{i,j}$ exactly when it crosses sides $s_i$ and $s_j$ of the $k$-gon. Then, similarly to the proof of Lemma 13, the lines in a single class visible to sides $s_x$ with $i < x < j$ can be treated as a single pseudoline, visible only on one side. And the lines visible to sides $s_x$ with $x < i$ or $j < x$ can be treated as another pseudoline, also visible on a side.

The number of corners on these pseudolines is at most $|C_{i,j}|$. The pseudoline can be divided up into a sequence of segments, linearly ordered by which $k$-gon side they are visible from; there may be gaps where the pseudoline is not visible to any side, but these can be assigned arbitrarily to either neighboring side. Then each side that sees the pseudoline sees a number of the lines composing it equal to one plus the number of corners in the segment assigned to that side. Therefore each corner adds at most one to the complexity of $V(n,k)$.

Now given two such pseudolines, as in Lemma 15, there is only one side that can be visible to both of them, and we can apply lemma 14 with $a = 1$ and $m = 2\binom{k}{2}$. Adding the pseudoline-side visibilities to the number of corners on each pseudoline gives a total bound of

$$V(n,k) \leq \sum_{i=1}^{k}\sum_{j=1}^{k} |C_{i,j}| + 2\binom{k}{2} + \binom{k}{2} \leq n + (3/2)k^2 - k.$$

■

**Theorem 3.** *The complexity of the cells on the inside of a $k$-gon, cut by $n \geq 1$ lines, is at most $11n + (3/2)k^2$.* ■

This gives the tightest known bound for fixed $k$, of $11n+O(1)$. However the best construction known is Figure 6 (appropriately modified to fit in the $k$-gon), which has complexity $10n+O(1)$.

**Open Problem 2.** *Reduce the gap between the lower bound of $10n + O(1)$ and the upper bound of $11n + O(1)$ on the complexity of the cells on the inside of a $k$-gon when $k$ is a fixed constant.*

## 5. Recursive Bounds for Polygons

In this section we show that the maximum complexity of the zone of a $k$-gon, for $k$ that is $O(n)$, is $O(n\alpha(n,k))$. In the final section of this paper we show that our bound is tight for all $n$ and $k$, up to constant factors. We have recently learned that the same $O(n\alpha(n,k))$ bound can be obtained in a conceptually simpler way. First observe that the sequence of lines counted in $V(n,k)$ forms an $n$-letter Davenport-Schinzel sequence of order three; that is, there can be no embedded $a\ldots b\ldots a\ldots b\ldots a$. Moreover, this sequence can be divided into $k$ contiguous blocks, such that each block contains no repeated letters. Sharir [7] has shown an $O(n\alpha(n,k))$ bound on the length of such a Davenport-Schinzel sequence. In fact, his proof also simplifies the previous $O(n\alpha(n))$ upper bound argument for arbitrary order-3 Davenport-Schinzel sequences [5].

Our upper bound of $O(n\alpha(n,k))$ depends on the following recurrence bounding $V(n,k)$. (Sharir's proof depends on a similar, but slightly less complicated, recurrence.)

**Lemma 17.** *Let $k \leq b \cdot k'$. Then*

$$V(n,k) \leq \max\ \left\{ 2n_0 + V(n_0+b,b) + \sum_{i=1}^{b} V(n_i+2k',k'+1) \ \middle|\ \sum_{i=0}^{b} n_i = n \right\}.$$

Figure 9. Cutting a $k$-gon into one $b$-gon and $b$ $(k'+1)$-gons.

**Proof:** Our overall strategy is to conceptually cut the $k$-gon into a central $b$-gon along with $b$ polygons around this $b$-gon, each with no more than $k'+1$ sides. We then define a new pseudoline arrangement within each of these polygons, such that the total number of visible line-side pairs in the new arrangements is an upper bound on $V(n,k)$, the original number of visible line-side pairs in the $k$-gon.

Choose a configuration of $n$ lines in a $k$-gon achieving the maximum value of $V(n,k)$ pairs. Bundle contiguous sides of the $k$-gon into $b$ groups, with at most $k'$ sides in each group. Define $n_i$, for $1 \le i \le b$, to be the number of lines in the arrangement having both ends within group $i$; let $n_0$ be the number of lines having ends in two groups. Then $\sum_{i=0}^{b} n_i = n$.

We will count the number of visibilities in the original arrangement by forming new arrangements for each group of sides. Consider the lines visible to a particular group $i$. These lines fall into four possible types: (1) lines starting and ending in group $i$, (2) lines starting in group $i$ and ending in another group, (3) lines starting and ending in two other groups, and (4) lines starting and ending in the same other group.

The lines of type (1) will remain unchanged in the new arrangement for group $i$.

We subdivide the lines of type (2) according to which side of group $i$ they cross. As in Lemma 16, we can form the lines crossing a single side into two pseudolines in the new arrangement for group $i$. As in that lemma, we must also count the number of corners formed on those pseudolines. The total number of pseudolines formed in group $i$ is $2k'$; however two of those, the first and last pseudolines in the clockwise order around group $i$, cannot create any visibilities that are counted in $V(n,k)$. Therefore the number of pseudolines contributing to the visibilities within group $i$ is $2k'-2$. The number of corners in all these pseudolines is at most equal to the number of lines of type (2); the sum of these numbers over all groups is $2n_0$.

We subdivide the lines of type (4) according to which other group they belong to. The lines from a single group $j$ can be treated as a single pseudoline of type (3) as viewed from sides in group $i$. However, we must also count the corners on this pseudoline. These corners are counted (for all choices of $i$ at once) by adding another side to group $j$ to form a closed polygon with no more than $k'+1$ sides. See Figure 9.

The lines and new pseudolines of type (3) can be treated as a single pseudoline "ceiling" when counting visibilities from group $i$; however, we must also count the number of corners on that ceiling. This is just the number of lines and pseudolines of type (3) composing the ceiling, which can be counted by creating a new arrangement consisting of all lines and pseudolines of type (3), cut by a $b$-gon corresponding to the $b$ groups of sides. (For a geometric realization, the $b$-gon may have to bend out and exchange an endpoint with a new pseudoline as in Figure 9.) The number of line-side visibilities in this $b$-gon arrangement counts the corners on all ceilings at once. (In fact, this slightly overcounts since only one side of each new pseudoline is visible.) Thus the total number of corners for all groups is at most $V(n_0 + b, b)$.

Summarizing, we have $V(n_i + 2k', k' + 1)$ visibilities within group $i$ and corners on the pseudoline for the type (4) lines contributed by group $i$ to visibilities in other groups. We have $2n_0$ corners on all the pseudolines created for the type (2) visibilities. And we have $V(n_0+b, b)$ corners on the pseudolines created for the type (3) visibilities and the pseudolines of type (3) created to count the type (4) visibilities. Adding these together gives the total bound. ■

Now let us define an Ackermann function $A(i, j)$ as follows:

- $A(1, j) = 6j$.
- $A(i, j) = A(i - 1, j + 1)$ if $j \leq 6$.
- $A(i, j) = A(i - 1, A(i, j - 6))$ if $j > 6$.

Let $\alpha_i(x) = \max \{1\} \cup \{j \geq 1 \mid A(i, j+1) \leq x\}$ and $\alpha(x, y) = \min \{i \geq 1 \mid A(i, \lfloor x/y \rfloor) \geq x\}$. Our Ackermann function is somewhat nonstandard; at the end of the paper we show that nevertheless $\alpha(x, y)$ is within a constant factor of other inverse Ackermann functions [8, 9]. Our functions have the following properties:

**Lemma 18.** *For any $k \geq 0$ and $\ell \geq 0$ with $k + l \geq 1$, $A(i, j) > A(i - k, j - \ell)$.*

**Proof:** We prove the lemma by induction; to prove it for $(i, j)$ we assume that it holds for all $(i', j')$ with $i' < i$ or with $i' = i$ and $j' < j$. Then we show that $A(i, j) > \max\{A(i - 1, j), A(i, j - 1)\}$; the full lemma for $(i, j)$ easily follows.

- For $i = 1$, $A(i, j) > A(i, j - 1)$ immediately from the definition.
- For $j \leq 6$, $A(i, j) = A(i - 1, j + 1) > A(i - 1, j) = A(i, j - 1)$.
- For $j = 7$, $A(i, j) = A(i - 1, A(i, 1)) > A(i - 1, A(2, 1)) = A(i - 1, 12) > A(i - 1, 7) = A(i, j - 1)$.
- For $i = 2$ and $j = 7$, $A(i, j) = 72 > 42 = A(i - 1, j)$.
- For $i = 2$ and $j > 7$, $A(i, j) = 6(A(i, j - 6)) > 6(A(i - 1, j - 6)) = 36(j - 6) > 6j = A(i - 1, j)$.

In all the remaining inequalities to check, both left and right sides follow the recursive definition. Then $A(i, j) = A(i - 1, A(i, j - 6)) > A(i - 1, A(i, j - 7)) = A(i, j - 1)$ and $A(i, j) = A(i - 1, A(i, j - 6)) > A(i - 2, A(i - 1, j - 6)) = A(i - 1, j)$. ■

**Corollary 1.** *For any $i$ and $x$, $\alpha_i(x) \leq \alpha_{i-1}(x)$.*

**Lemma 19.** *If $i \geq 2$, then $\alpha_i(x) < \sqrt{x}$.*

**Proof:** We prove the lemma for $i = 2$; the remaining cases follow from Corollary 1. If $x \leq 72$, then $\alpha_2(x) \leq 6$, and the truth of the lemma can be seen by inspection of the possible cases. Otherwise, using induction, $\alpha_2(x) = 1 + \alpha_2(x/6) < 1 + \sqrt{x/6} < \sqrt{x}$. ∎

**Lemma 20.** *For any $i > 1$ and $x$, $\alpha_i(\alpha_{i-1}(x) + 1) \leq \max\{6, \alpha_i(x) - 6\}$.*

**Proof:** If $\alpha_i(x) \leq 6$, then $\alpha_i(\alpha_{i-1}(x) + 1) \leq 6$. Otherwise,

$$A(i, \alpha_i(\alpha_{i-1}(x) + 1) + 6 + 1) = A(i - 1, A(i, \alpha_i(\alpha_{i-1}(x) + 1) + 1)) \leq A(i - 1, \alpha_{i-1}(x) + 1) \leq x.$$

But for any $y$, if $A(i, y + 1) \leq x$, it follows that $y \leq \alpha_i(x)$. Hence $\alpha_i(\alpha_{i-1}(x) + 1) + 6 \leq \alpha_i(x)$. ∎

**Lemma 21.** *There is a constant $c \geq 3$ such that for any $i < \alpha(n, k)$, $V(n, k) \leq 2in + 3ik\alpha_i(k) + ci(k - 3)$.*

**Proof:** For $i = 1$, the lemma gives the bound of Lemma 15 along with a term linear in $k$. So let $i > 1$. If $k \leq \sqrt{n}$, then the lemma follows from Lemma 15. Assume that $\alpha_i(k) < 12$. Then $A(i, 12) \geq k$ and $A(i, 13) \geq k^2 \geq n$. So the one-variable function $\alpha(n)$ is $O(i)$. Now $i < \alpha(n, k)$ implies that $A(i, \lfloor n/k \rfloor) < n$, which means that $n/k < 13$. By choosing $c$ large enough, the lemma then follows from the Davenport-Schinzel bound of $n\alpha(n)$ [5].

So we may assume $i > 1$ and $\alpha_i(k) \geq 12$. Let $k' = \alpha_{i-1}(k)$, and let $b = \lceil k/k' \rceil$.

Now $A(i, k) < n$. If $k' = 1$, then $\alpha_i(k) = 1 = \alpha_{i-1}(k)$, and the bound follows easily from that for $i - 1$. Otherwise, note that $k' < b$ (by Lemma 19), $b \leq (n + 1)/2$, and

$$b\alpha_{i-1}(b) \leq bk' \leq k + k' \leq k + b.$$

By Lemma 17,

$$V(n, k) \leq 2n_0 + V(n_0 + b, b) + \sum_{j=1}^{b} V(n_j + 2k', k' + 1).$$

We inductively use the bound we are proving with $i - 1$ for $V(n_0, b)$ and with $i$ for $V(n_j + 2k', k' + 1)$:

$$\begin{aligned} V(n,k) \leq\ & 2n_0 + 2(i-1)(n_0 + b) + 3(i-1)b\alpha_{i-1}(b) + c(i-1)(b-3) \\ & + \sum_{j=1}^{b}\Big(2i(n_j + 2k') + 3i(k'+1)\alpha_i(k'+1) + ci(k'-2)\Big). \end{aligned}$$

We now gather terms involving $n_0$ and $n_j$,

$$\begin{aligned} V(n,k) \leq\ & 2in + 2(i-1)b + 3(i-1)b\alpha_{i-1}(b) + c(i-1)(b-3) \\ & + 4ibk' + 3ib(k'+1)\alpha_i(k'+1) + cib(k'-2), \end{aligned}$$

and apply the facts that $b\alpha_{i-1}(b) \leq k + b$ and $bk' \leq k + b$,

$$V(n, k) \leq 2in + 9ib + 7ik + 3i(k + 2b)\alpha_i(k' + 1) + ci(k - 3).$$

By Lemma 20 and the fact that $\alpha_i(k) \geq 12$,

$$\begin{aligned} V(n,k) &\leq 2in + 9ib + 7ik + 3i(k + 2b)(\alpha_i(k) - 6) + ci(k - 3) \\ &\leq 2in - 27ib - 11ik + 6ib\alpha_i(k) + 3ik\alpha_i(k) + ci(k - 3) \\ &\leq 2in + 3ik\alpha_i(k) + ci(k - 3), \end{aligned}$$

since by Corollary 1, $\alpha_i(k) \leq k' \leq (k + b)/b$. ■

**Theorem 4.** *The complexity of the cells on the inside of a $k$-gon, cut by $n$ lines, is $O(n\alpha(n, k))$.* ■

# 6. Lower Bounds for Polygons

Wiernik gave a construction of $n$ line segments with lower envelope complexity $\Omega(n\alpha(n))$ [10]. By placing such a collection of segments inside a convex $n$-gon and extending segments to the $n$-gon with curves, we can form a pseudoline arrangement with zone complexity $\Omega(n\alpha(n))$. In this section we show how to generalize this construction for any $k$, giving a lower bound of $\Omega(n\alpha(n, k))$.

We adapt Shor's construction for the lower-envelope complexity of line segments [8], a simplification of the construction due to Wiernik [10]. As our lower bound construction only works for pseudolines, we need not concern ourselves with some of the details of Shor's construction, such as the careful handling of line segment slopes.

Shor constructs an arrangement of positive-slope line segments $S(i, j, r)$ with nonlinear lower-envelope complexity as follows. First define an Ackermann function $F(i, j)$ by $F(1, j) = 1$, $F(i, 1) = 2 \cdot F(i - 1, 2)$, and $F(i, j) = F(i, j - 1) \cdot F(i - 1, F(i, j - 1))$. Arrangement $S(i, j, r)$ contains $F(i, j)$ $j$-*fans*. A $j$-fan is a set of $j$ line segments that share the same left endpoint. Within each $j$-fan the segments can be numbered $1, 2, \ldots j$, such that the slope of each segment is at least $r$ times the slope of the preceding one, where $r$ is a real number at least 3.

Arrangement $S(i, j, r)$ is defined recursively. $S(1, j, r)$ is a single $j$-fan. $S(i, 1, r)$ is constructed from $S(i - 1, 2, r)$ by translating the larger-slope segment of each 2-fan by a tiny distance $\epsilon$, so that the smaller-slope segment has endpoint somewhat to the left of the larger-slope segment. Each segment is then a 1-fan and the lower-envelope complexity doubles.

In the general inductive step, we generate many copies of $S(i, j - 1, r)$ and a single copy of $S^* = S(i - 1, F(i, j - 1), r^*)$, where $r^*$ is determined by the geometry of $S(i, j - 1, r)$. Arrangement $S^*$ is flattened and then tilted by an affine transformation so that all slopes of segments in $S^*$ are very close to 1. This transformation leaves the lower envelope complexity unchanged. There will be some tiny $\epsilon > 0$ such that if each segment of $S^*$ is translated a distance no greater than $\epsilon$, then the only changes in the lower envelope occur at the left endpoints of fans.

Next each copy of $S(i, j - 1, r)$ is first flattened so that all slopes are less than $1/r$ and then shrunk to be smaller than $\epsilon$. We place each copy of $S(i, j - 1, r)$ next to a fan left endpoint of $S^*$, and then perturb the segments of $S^*$ so that one segment of $S^*$ joins each $(j - 1)$-fan

Figure 10. Pseudoline arrangement with nonlinear zone complexity.

in the nearby copy of $S(i, j-1, r)$. The trick is that segments of $S^*$ are perturbed so that the largest-slope segment of each fan will have rightmost left endpoint, next largest next rightmost, and so on. With suitable choice of $r^*$, each $F(i, j-1)$-fan emerges from its copy of $S(i, j-1, r)$ and then forms a "caustic curve", that is, an arrangement with $F(i, j-1)$ subsegments in its lower envelope. This caustic curve lies to the right of all right endpoints of segments in the copy of $S(i, j-1, r)$. See Figure 10.

The complexity of the lower envelope of $S(i, j, r)$ is the sum of the complexities of $F(i-1, F(i, j-1))$ copies of $S(i, j-1, r)$, one copy of $S^*$, and $F(i-1, F(i, j-1))$ caustic curves of complexity $F(i, j-1)$. The solution is $ijF(i, j)$ as confirmed by the following equality:

$$\begin{aligned} ijF(i,j) \;&=\; F(i-1, F(i,j-1)) \cdot i(j-1)F(i,j-1) \;+ \\ &\quad\; (i-1)F(i,j-1) \cdot F(i-1, F(i,j-1)) \;+\; F(i-1, F(i,j-1)) \cdot F(i,j-1). \end{aligned}$$

We now show how to extend the segments of $S(i, j, r)$ to pseudolines and fit them into a polygonal chain with $F(i, j)+1$ sides, that is, one side for each fan plus one final side. One or two additional sides can be added at the end to close the polygon. First, $S(1, j, r)$ fits into a chain with 2 sides. The left endpoint lies on a horizontal side, and right endpoints are extended with curves to intersect a second side with arbitrary positive slope. The curves are such that they do not intersect each other. Second, $S(i, 1, r)$ fits into a chain with $F(i, 1) + 1 = 2F(i-1, 2) + 1$ sides by simply subdividing each edge, except the last, of the chain holding $S(i-1, 2, r)$, so that each fan starts on its own side. In the general step, we need $F(i-1, F(i, j-1))$ copies of chains with $F(i, j-1) + 1$ sides to hold all the copies of $S(i, j-1, r)$. The last side of a chain for each copy doubles as the first side of the chain for the next copy, as shown in Figure 10. One final side is added to receive the extensions of the segments of $S^*$.

We now confirm that for each choice of the number of pseudolines $n$ and the number of polygon sides $k$, this construction matches the upper bound of the last section. Define an inverse Ackermann function $\phi(n, k) = \min\{i \geq 1 \mid (n/k)F(i, \lfloor n/k \rfloor) \geq n\}$. The construction above gives zone complexity $ijF(i, j)$ for an arrangement of $jF(i, j)$ pseudolines and a polygon with $F(i, j) + O(1)$ sides. Setting $j = \lfloor n/k \rfloor$ and $i = \phi(n, k)$, the construction gives zone complexity $n\phi(n, k)$ for an arrangement of $n$ pseudolines in a $(k + O(1))$-gon. We now show that for $n/k \geq 7$, $\phi(n, k)$ is within a constant of $\alpha(n, k)$, the function defined in the last section. These inverse Ackermann functions are also within constants of more usual inverse Ackermann functions, for example the one used by Tarjan [9].

**Lemma 22.** *If $n/k \geq 7$, $\phi(n,k) - 1 \leq \alpha(n,k) \leq 4 \cdot \phi(n,k)$.*

**Proof:** For each choice of $i$ and $j$, $F(i+1,j) \geq A(i,j)$, so $\alpha(n,k) \geq \phi(n,k) - 1$. For each $i$ and each $j \geq 7$, $A(2i,2j) \geq A(2i,j+7) \geq jF(i,j)$, so $(1/2)\alpha(n,k) \leq \phi(n,2k) \leq 2\phi(n,k)$. ■

**Theorem 5.** *The maximum complexity of the cells on the inside of a $k$-gon, cut by $n$ pseudolines, is $\Theta(n\alpha(n,k))$.*

**Proof:** This follows from Theorem 4 along with Lemma 22 for $n \geq 7k$ and the $n\alpha(n)$ lower bound for $n < 7k$. ■

Both Wiernik's construction and our lower bound construction use pseudolines, and it seems difficult to modify them to use straight lines instead. We close by posing the following open problem.

**Open Problem 3.** *Find a construction for a straight line arrangement cut by a polygon having superlinear zone complexity, or show that no such construction exists.*

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