An Eigenvalue-less Eigenvector Algorithm. The eigenvectors of the real symmetric matrix

$$M = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$$

are the real solutions (r, s, t) for the system of equations

$$\begin{aligned} r^2(bs + ct) + r(s^2(d - a) + t^2(f - a) + 2est) - (s^2 + t^2)(bs + ct) &= 0 \\ s^2(br + et) + s(r^2(a - d) + t^2(f - d) + 2crt) - (r^2 + t^2)(br + et) &= 0 \\ t^2(cr + es) + t(r^2(a - f) + s^2(d - f) + 2brs) - (r^2 + s^2)(cr + es) &= 0. \end{aligned}$$

To see why this is true, make the following substitutions into the max-min characteristic polynomials. $XX = \frac{1}{2}(-a + d + f)$, $YY = \frac{1}{2}(a - d + f)$, $ZZ = \frac{1}{2}(a + d - f)$, $XY = -b$, $XZ = -c$, $YZ = -e$. Note that $YY + ZZ = a$, $XX + ZZ = d$, $XX + YY = f$. Then observe that every (necessarily real) eigenvalue of a real symmetric matrix is associated with a real eigenvector.

This note is dedicated to Gus Watts (1916–2000), a retired aeronautical engineer, who died shortly after crafting for the author a tetrahedron of solid poplar through which holes were bored by a specially designed non-wandering drill bit so that the tetrahedron could rotate freely about a rod through any of its principal axes.

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Tangent Spheres and Triangle Centers

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1. TANGENT SPHERES. Any four mutually tangent spheres determine six points of tangency. We say that a pair of tangencies $\{t_i, t_j\}$ is *opposite* if the two spheres determining t_i are distinct from the two spheres determining t_j. Thus the six tangencies are naturally grouped into three opposite pairs, corresponding to the three ways of partitioning the four spheres into two pairs. Altshiller-Court [**1**, §630, p. 231] proved the following result about these opposite pairs, which we use to define two new triangle centers.

Lemma 1. *The three lines through opposite points of tangency of any four mutually tangent spheres in* $\mathbb{R}^3$ *are coincident.*

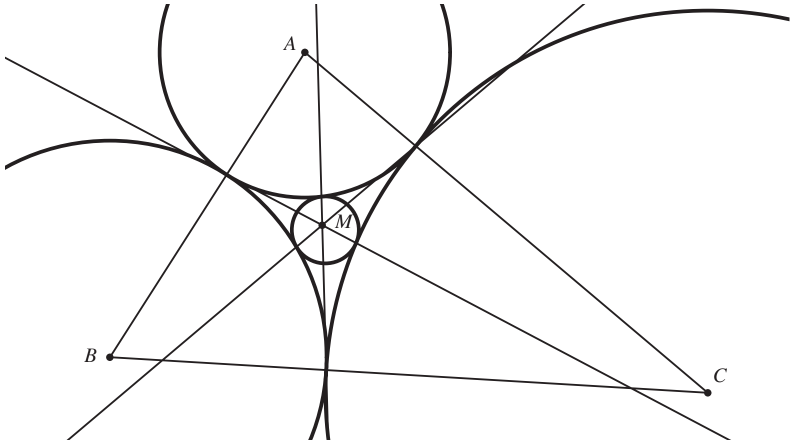

Figure 3. A triangle ABC and its new center M.

Proof. If three spheres have a common tangency, the three lines all meet at that point; otherwise, each sphere either contains all of or none of the other three spheres. Let the four given spheres S_i ($i \in \{1, 2, 3, 4\}$) have centers $\bar{x}_i$ and radii r_i. If S_i contains none of the other spheres, let $R_i = r_i^{-1}$, else let $R_i = -r_i^{-1}$. Then the point of tangency t_{ij} between spheres S_i and S_j can be expressed in terms of these values as

$$t_{ij} = \frac{R_i}{R_i + R_j}\bar{x}_i + \frac{R_j}{R_i + R_j}\bar{x}_j.$$

This is a weighted average of the two sphere centers, with weights inversely proportional to the (signed) radii.

Now consider the point

$$M = \frac{\sum_{i=1}^{4} R_i \bar{x}_i}{\sum_{i=1}^{4} R_i}$$

formed by taking a similar weighted average of all four sphere centers. Then

$$M = \frac{R_1 + R_2}{(R_1 + R_2) + (R_3 + R_4)} t_{12} + \frac{R_3 + R_4}{(R_1 + R_2) + (R_3 + R_4)} t_{34},$$

i.e., M is a weighted average of the two tangencies t_{12} and t_{34}, and therefore lies on the line $t_{12}\,t_{34}$. By a symmetric argument, M also lies on line $t_{13}\,t_{24}$ and line $t_{14}\,t_{23}$, so these three lines are coincident. ■

A similar weighted average for three mutually externally tangent circles in the plane gives the Gergonne point of the triangle formed by the circle centers. Altshiller-Court's proof is based on the fact that the lines $\bar{x}_i\,t_{ij}$ meet in triples at the Gergonne points of the faces of the tetrahedron formed by the four sphere centers. We need the following special case of the lemma in which the four sphere centers are coplanar:

Corollary 1. *The three lines through opposite points of tangency of any four mutually tangent circles in $\mathbb{R}^2$ are coincident.*

2. NEW TRIANGLE CENTERS. Any triangle ABC uniquely determines three mutually externally tangent circles centered on the triangle vertices; if the triangle's sides have length a, b, c then these circles have radii $(-a+b+c)/2$, $(a-b+c)/2$, and $(a+b-c)/2$. The sides of triangle ABC meet its incenter at the three points of tangency of these circles.

For any three such circles O_A, O_B, O_C, there exists a unique pair of circles O_S and $O_{S'}$ tangent to all three. The quadratic relationship between the radii of the resulting two quadruples of mutually tangent circles was famously memorialized in Frederick Soddy's poem, "The Kiss Precise".

The set $\mathbb{R}^2 \setminus (O_A \cup O_B \cup O_C)$ has five connected components, three of which are disks and the other two of which are three-sided regions bounded by arcs of the three circles; we distinguish O_S and $O_{S'}$ by requiring O_S to lie in the bounded three-sided region and $O_{S'}$ to lie in the unbounded region. Note that O_S is always externally tangent to all three circles, but $O_{S'}$ may be internally or externally tangent depending on the positions of points ABC. If O_A, O_B, and O_C have a common tangent line, then we consider $O_{S'}$ to be that line, which we think of as an infinite-radius circle intermediate between the internally and externally tangent cases.

We can then use Corollary 1 to define two triangle centers: let M denote the point of coincidence of the three lines $t_{AS}\, t_{BC}$, $t_{BS}\, t_{AC}$, and $t_{CS}\, t_{AB}$ determined by the pairs of opposite tangencies of the four mutually tangent circles O_A, O_B, O_C, and O_S (Figure 3), and similarly let M' denote the point of coincidence of the three lines $t_{AS'}\, t_{BC}$, $t_{BS'}\, t_{AC}$, and $t_{CS'}\, t_{AB}$ determined by the pairs of opposite tangencies of the four mutually tangent circles O_A, O_B, O_C, and $O_{S'}$. The definitions of M and M' do not depend on the ordering of the vertices nor on the scale or position of the triangle.

Despite their simplicity of definition, and despite the large amount of work that has gone into triangle geometry (see [**2**] and [**3**]), the centers M and M' do not appear in the lists of over 400 known triangle centers collected by Clark Kimberling and Peter Yff (personal communications).

3. RELATIONS TO KNOWN CENTERS. M and M' are not the only triangle centers related to the Soddy circles O_S and $O_{S'}$. The centers S and S' of the Soddy circles are known (see [**4**] or [**5**]); S is also the point of coincidence of the three lines $A\, t_{AS}$, $B\, t_{BS}$, $C\, t_{CS}$ and similarly for S'. The Gergonne point Ge can be defined in a similar way as the point of coincidence of the three lines $A\, t_{BC}$, $B\, t_{AC}$, and $C\, t_{AB}$. It is known that S and S' are collinear with and harmonic to Ge and I, where I denotes the incenter of triangle ABC [**5**]. Similarly Ge and I are collinear with and harmonic to the isoperimetric point and the point of equal detour [**6**].

Theorem 1. *M and M' are collinear with and harmonic to Ge and I.*

Proof. By using ideas from our proof of Lemma 1, we can express M as a weighted average of S and Ge:

$$
\begin{aligned}
M &= \frac{R_A A + R_B B + R_C C}{R_A + R_B + R_C + R_S} + \frac{R_S S}{R_A + R_B + R_C + R_S} \\
&= \frac{R_A + R_B + R_C}{R_A + R_B + R_C + R_S} Ge + \frac{R_S}{R_A + R_B + R_C + R_S} S.
\end{aligned}
$$

Hence, M is collinear with S and Ge. Collinearity with Ge and I follows from the known collinearity of S with Ge and I. A symmetric argument applies to M'.

We omit the proof of harmonicity, which we obtained by manipulating trilinear coordinates of the new centers in *Mathematica*. See `http://www.ics.uci.edu/~eppstein/junkyard/tangencies/trilinear.pdf` for the detailed calculations. ■

A simple compass-and-straightedge construction for the Soddy circles and our new centers M and M' can be derived from the following further relation:

Theorem 2. *Let ℓ_A denote the line through point A, perpendicular to the opposite side BC of the triangle ABC. Then the two lines ℓ_A and $t_{AS}\,t_{BC}$ and the circle O_A are coincident.*

Proof. Let O_D be a circle centered at t_{BC}, such that O_A and O_D cross at right angles. Then inverting through O_D produces a figure in which O_B and O_C have been transformed into lines parallel to ℓ_A, while O_A is unchanged. Since the image of O_S is tangent to O_A and to the two parallel lines, it is a circle congruent to O_A and centered on ℓ_A. Therefore, the inverted image of t_{AS} is a point p where ℓ_A and O_A cross. Points t_{BC}, t_{AS}, and p are collinear since one is the center of an inversion swapping the other two. ■

Since ℓ_A, O_A, and t_{BC} are all easy to find, one can use this result to construct the line $t_{AS}\,t_{BC}$, and symmetrically the lines $t_{BS}\,t_{AC}$ and $t_{CS}\,t_{AB}$, after which it is straightforward to find O_S, S, and M. A symmetric construction exists for $O_{S'}$, S', and M'.

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Acyclic and Totally Cyclic Orientations in Planar Graphs

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Let G be a planar graph imbedded in the plane, and let G^* be its dual graph. The graph G^* has one vertex for every face of G, and for every edge e of G there is a corresponding edge e^* of G^* joining the two faces adjacent to e. If we draw each e^* so that it crosses only the edge e, then G^* becomes a planar graph; see Figure 4. It is well known that G and G^* have the same number of spanning trees; see [**2**] for a