Orthogonal Dissection into Few Rectangles

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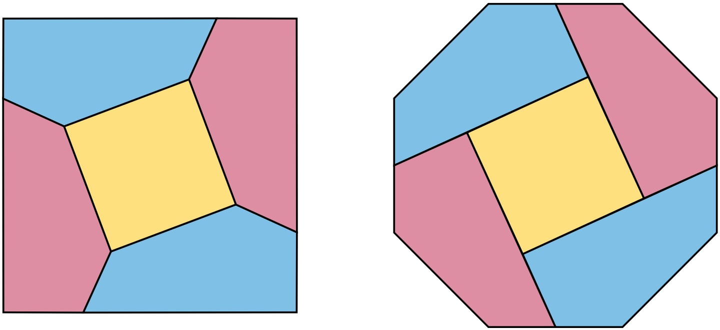
Dissection puzzles

Given two polygons of the same area
cut one into pieces and reassemble into the other

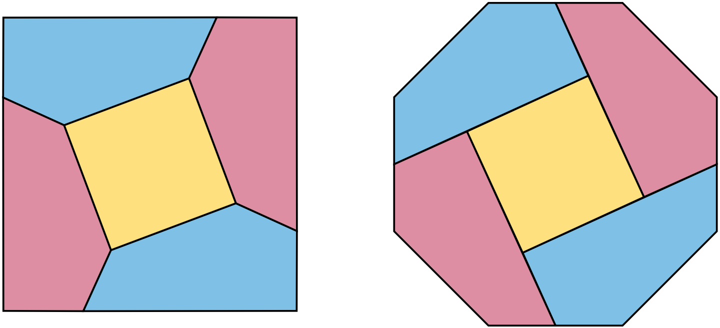


Typical puzzle: How few pieces are needed?

[Lindgren 1972; Frederickson 1997]

Wallace–Bolyai–Gerwien theorem

Equal-area polygons always have a dissection!



[Wallace and Lowry 1814; Bolyai 1833; Gerwien 1833]

David Hilbert, 1899

Unsatisfied by the rigor of Euclid's axioms for geometry

Taught a course on foundations of geometry, published his lecture notes as a book [Hilbert 1899]

Widely used and cited as a landmark in axiomatic geometry

Used dissections to axiomatize area

David Hilbert, 1900

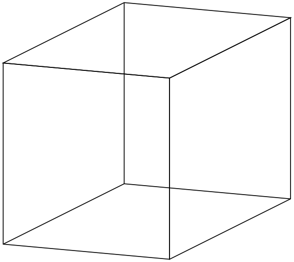
Presented ten unsolved problems for the following century in keynote address to International Congress of Mathematicians

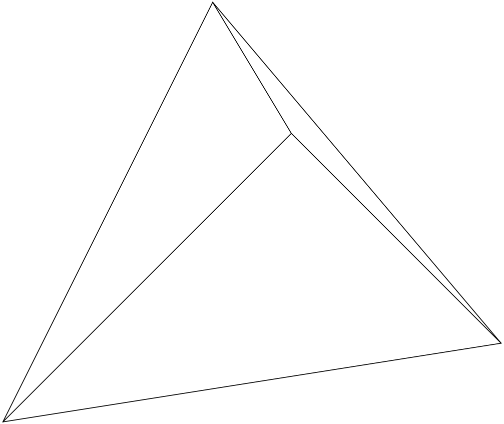
Later published as a list of 23 problems [Hilbert 1902]

Third problem: Do polyhedra of the same volume always have a dissection?

The first of Hilbert's problems to be solved

Cube and regular tetrahedron of same volume have no dissection





[Dehn 1901]

How did Dehn prove it?

Define a value (now called the *Dehn invariant*) associated with every polyhedron

Invariant: Does not change if you cut up and reassemble

Cube's value is zero, regular tetrahedron is nonzero

Later researchers:

Dissection exists if and only if volumes and invariants are equal [Sydler 1965]

Invariant = zero if and only if can dissect to space-filling polyhedron [Debrunner 1980]

Common impressions of Dehn invariants

It’s a number (FALSE!)

It’s a value in an infinite-dimensional tensor space, whatever that means (true but unhelpful)

The only useful thing to do with it is to compare it for equality with zero or other Dehn invariants (FALSE!)

Main ideas of this work

Work with only a finite input at a time,
instead of all possible inputs simultaneously

Make an arbitrary choice of basis

(Just like we do when computing in linear algebra)

$\Rightarrow$

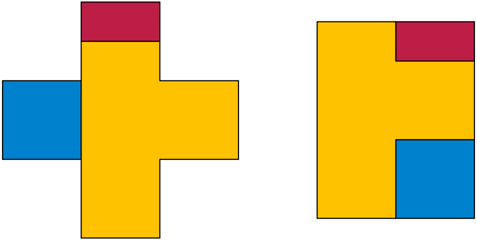
Dehn invariant becomes easier to understand:
it's just a matrix of rational numbers

We can do more than just comparing for equality:

Matrices have structure (their rank) that is
easy to compute and geometrically meaningful

A simpler problem with similar results

Dissection of 2D orthogonal polygons
by axis-parallel cuts and translation (no rotation!)



Rectangle-to-rectangle dissections can only scale vertical and horizontal dimensions by rational numbers [Dehn 1903]

(Because Greek cross can be dissected into a $2 \times 2\frac{1}{2}$ rectangle, it cannot also be dissected into a $\sqrt{5} \times \sqrt{5}$ square)

Proof idea: Dehn invariant!

Rational bases for irrational numbers

We need to handle coordinates that may be irrational numbers

Choose *basis*, system of real numbers $\{b_i\}$ s.t. all coordinates have a *unique* expression as a sum of rational multiples of b_i

This is just standard linear algebra over the field $\mathbb{Q}$

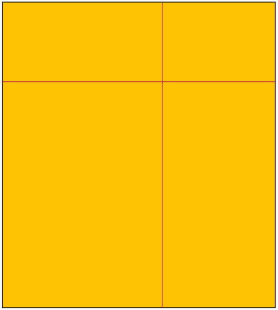
Handling all real numbers would need an uncountable Hamel basis but finitely many polygons can be handled with a finite basis

Choice of basis will not affect our results

The Dehn invariant of a rectangle

Express width, height in terms of basis (as vectors in $\mathbb{Q}^B$)

Dehn invariant = outer product



Width: $2 + \sqrt{2}$

Width vector: $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Height: $1 + 2\sqrt{2}$

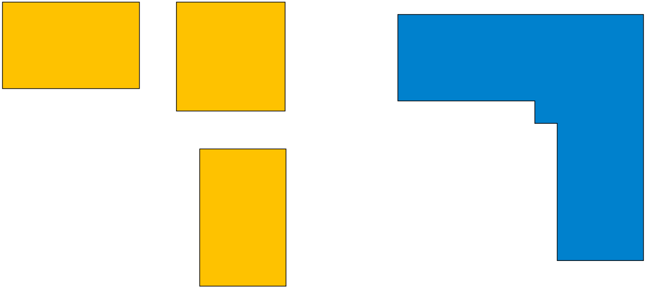
Height vector: $\begin{pmatrix} 1 & 2 \end{pmatrix}$

Outer product: $\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$

Outer products have rank = 1

The Dehn invariant of an orthogonal polygon

Decompose into rectangles (arbitrarily) and add their matrices



Example: union of $1 \times 2^{1/3}$, $2^{2/3} \times 2^{2/3}$, $2^{1/3} \times 1$ rectangles

For basis $\{1, 2^{1/3}, 2^{2,3}\}$, invariant is $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

Rank = length of shortest expression as a sum of outer products

$\leq$ number of rectangles in decomposition

Main results

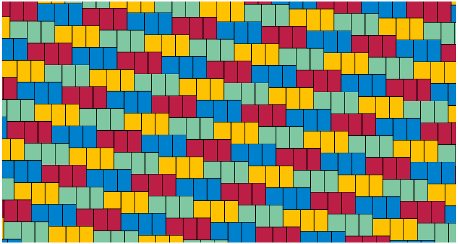
Dehn invariant is an invariant of orthogonal dissection
(easy but appears not to have been explicit for non-rectangles)

Area is linear function of matrix (also easy, simplifies next points)

Dissection exists if and only if Dehn invariants are equal

Every positive-area matrix comes from a union of rank-many rectangles, so rank = min # dissectable rectangles

Rank ≤ 2 if and only if polygon has dissection to plane-tiler



An open problem

For polyhedra with arbitrary cut directions & rotation, Dehn invariant is computed in the same way

- Use edge length and dihedral angle instead of width and height
- Drop the basis term coming from rational multiples of π in angles

Rank $\leq$ fewest edges of a polyhedron that the input can be dissected into

Is this an $O(1)$-approximation?

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