Möbius Transformations in Scientific Computing

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(including joint work with Marshall Bern from WADS'01 and SODA'03)

Outline

What are Möbius transformations?

Optimal Möbius transformation

Möbius-invariant constructions

Conclusions

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What are Möbius transformations?

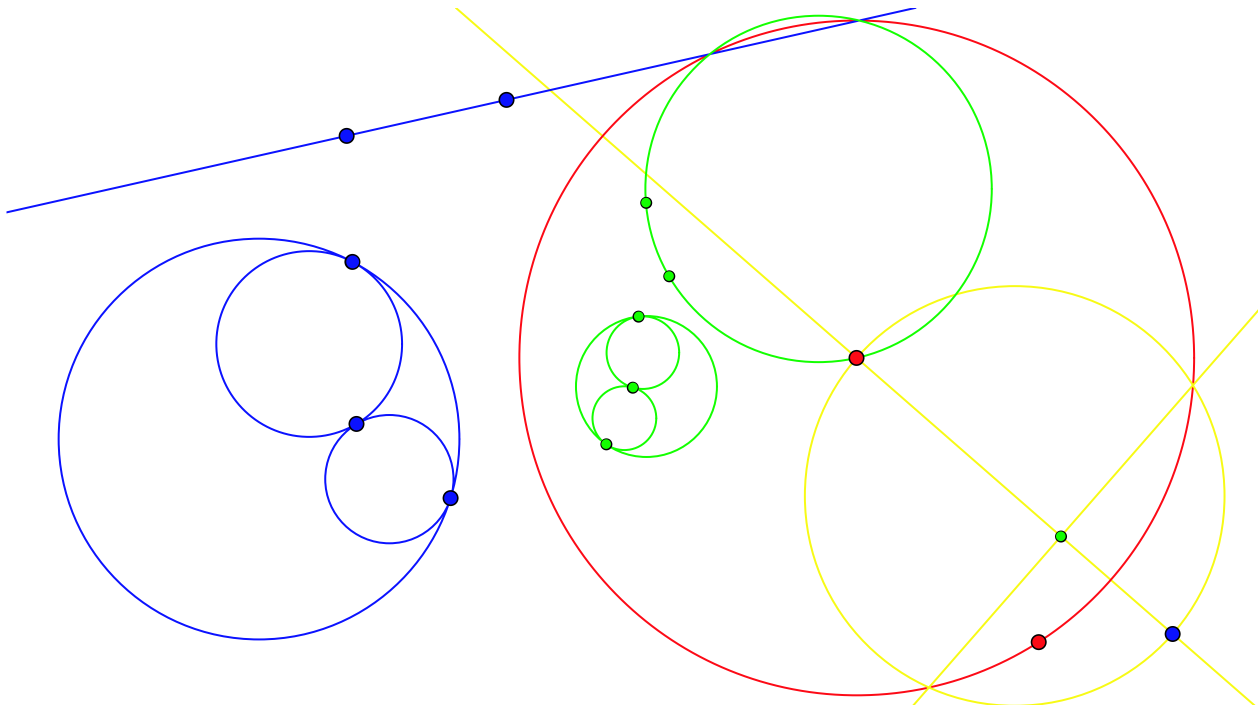
Fractional linear transformations of complex numbers:

$$z \rightarrow (a\,z + b)\ /\ (c\,z + d)$$

But what does it mean geometrically?
How to generalize to higher dimensions?
What is it good for?

Inversion

Given any circle (red below)
map any point to another point on same ray from center
product of two distances from center = radius^2



Circles ⇔ circles
(lines = circles through point at infinity)

Conformal (preserves angles between curves)

Möbius transformations = products of inversions

(or sometimes orientation-preserving products)

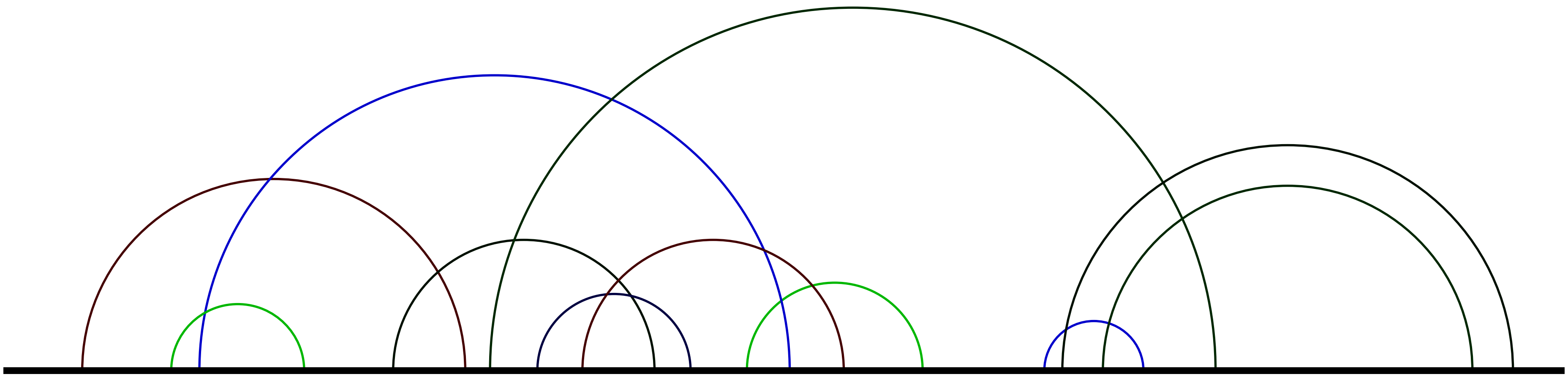
Forms group of geometric transformations

Contains all circle-preserving transformations

In higher dimensions (but not 2d) contains all conformal transformations

Hyperbolic interpretation of Möbius transformations

View plane as boundary of Poincaré (halfspace or unit disk) model of hyperbolic space



Möbius transformations of plane ↔ hyperbolic isometries

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Optimal Möbius transformation:

Given a planar (or higher dimensional) input configuration

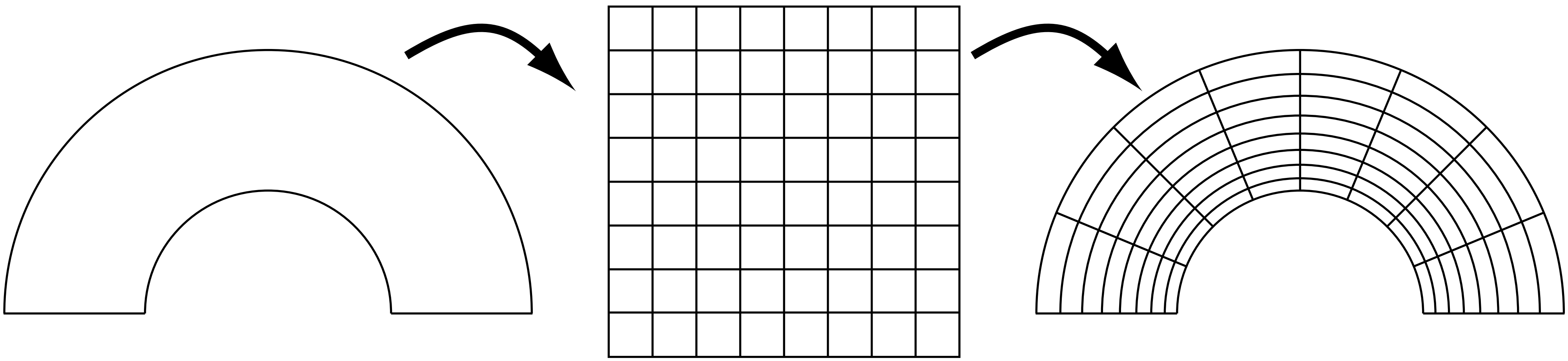
Select a Möbius transformation
from the (six-dimensional or higher) space of all Möbius transformations

That optimizes the shape of the transformed input

Typically min-max or max-min problems:
maximize min(set of functions describing transformed shape quality)

Application: conformal mesh generation

Given simply-connected planar domain to be meshed
Map to square, use regular mesh, invert map to give mesh in original domain



Different points of domain may have different requirements for element size
Want to map regions requiring small size to large areas of square

Conformal map is unique up to Möbius transformation

Optimization Problem:

Find conformal map maximizing min(size requirement * local expansion factor)
to minimize overall number of elements produced

Application: brain flat mapping [Hurdal et al. 1999]

Problem: visualize the human brain

All the interesting stuff is on the surface
But difficult because the surface has complicated folds

Approach: find quasi-conformal mapping brain → plane
Then can visualize brain functional units as regions of mapped plane

Avoids distorting angles but areas can be greatly distorted

As in mesh gen. problem, mapping unique up to Möbius transformation

Optimization problem:

Given map 3d triangulated surface → plane,
find Möbius transformation minimizing max(area distortion of triangle)

View Möbius transformation as choice of Poincaré model for hyperbolic space

Factor transformations into
choice of center point in hyperbolic model (affects shape)
Euclidean rotation around center point (doesn't affect shape)

Select optimal center point by quasiconvex programming

Represent solution quality as max of quasiconvex functions
where function argument is hyperbolic center point location
Hard part: proving that our objective functions are quasiconvex

Solve quasiconvex program [Amenta, Bern, Eppstein 1999]
Use center point location to find Möbius transformation

Result: find optimal transformation in linear time

(or more practically by simple hill climbing methods)

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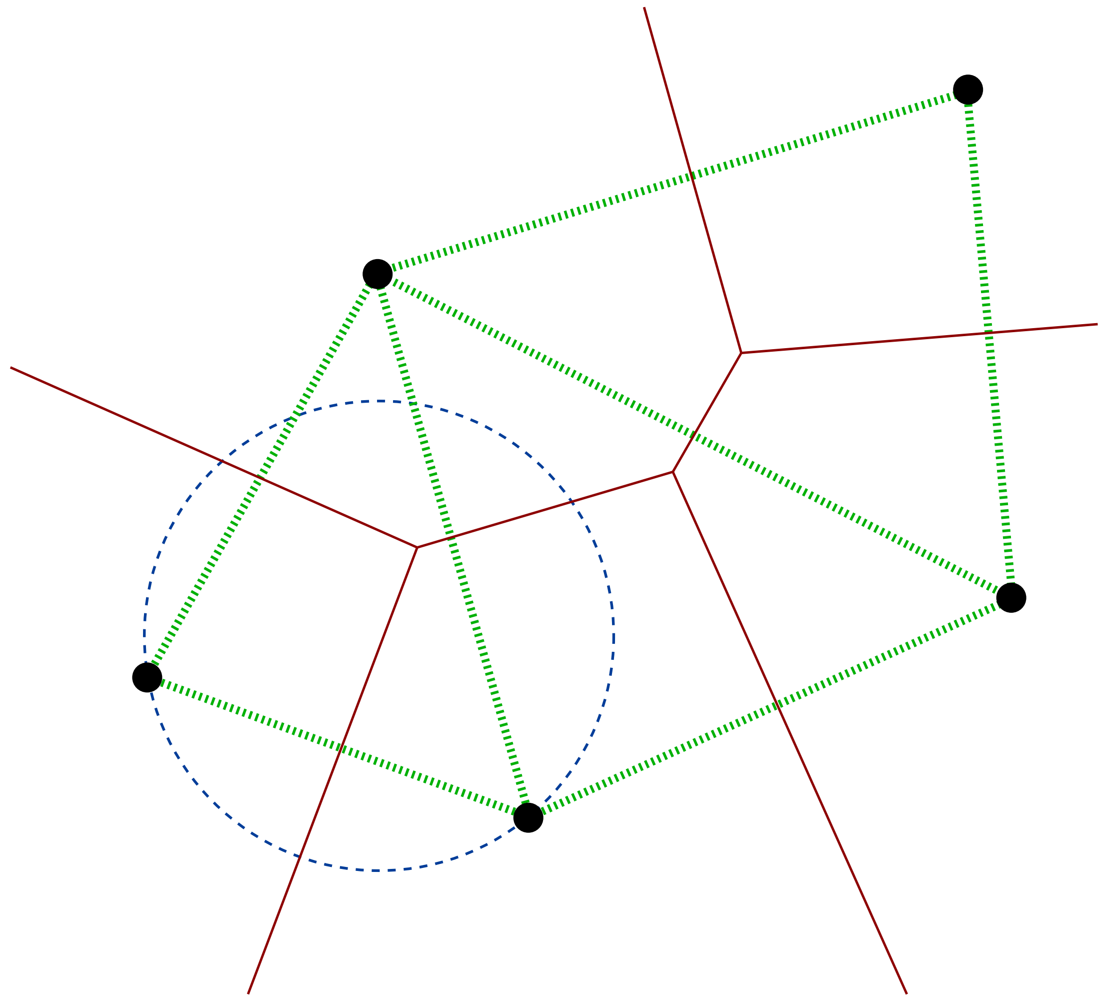
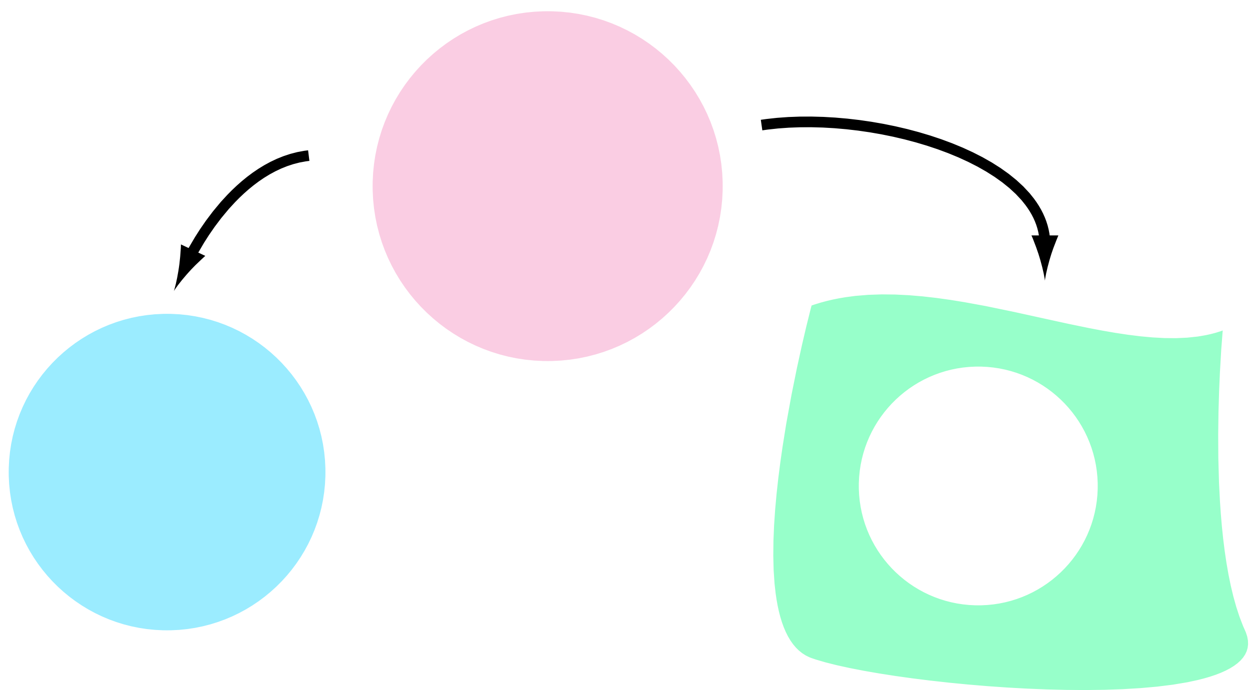
Delaunay triangulation

connect two points if both are on boundary of empty circle



Möbius transformation of Delaunay triangulation

Empty circle is transformed to another empty circle
or to empty complement of circle



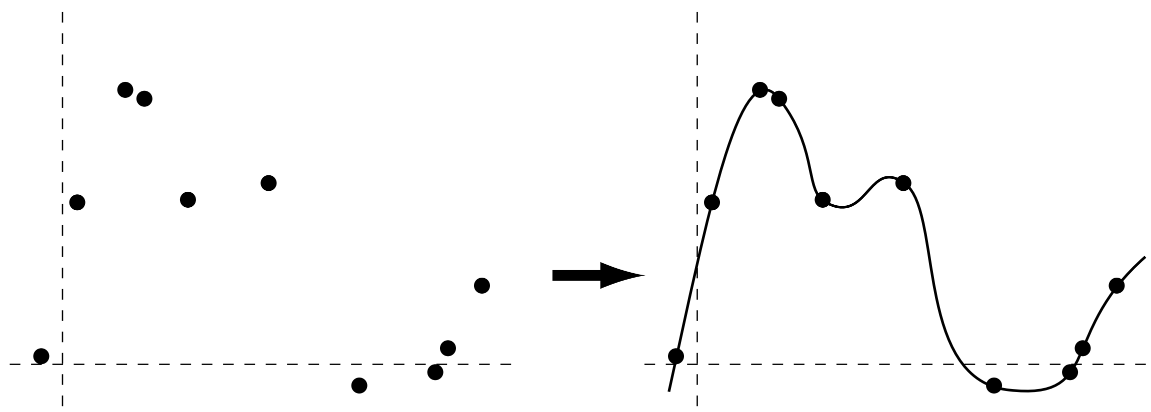
Augmented Delaunay triangulation:
connect points on boundaries of empty circles or co-circles
(equivalently, union of Delaunay and farthest-point Delaunay)

Invariant under Möbius transformation

Möbius-invariant interpolation?

Reconstruct a function (approximately)
given a discrete set of samples
(function values at finitely many data points)

Should exactly fit samples, be well behaved elsewhere

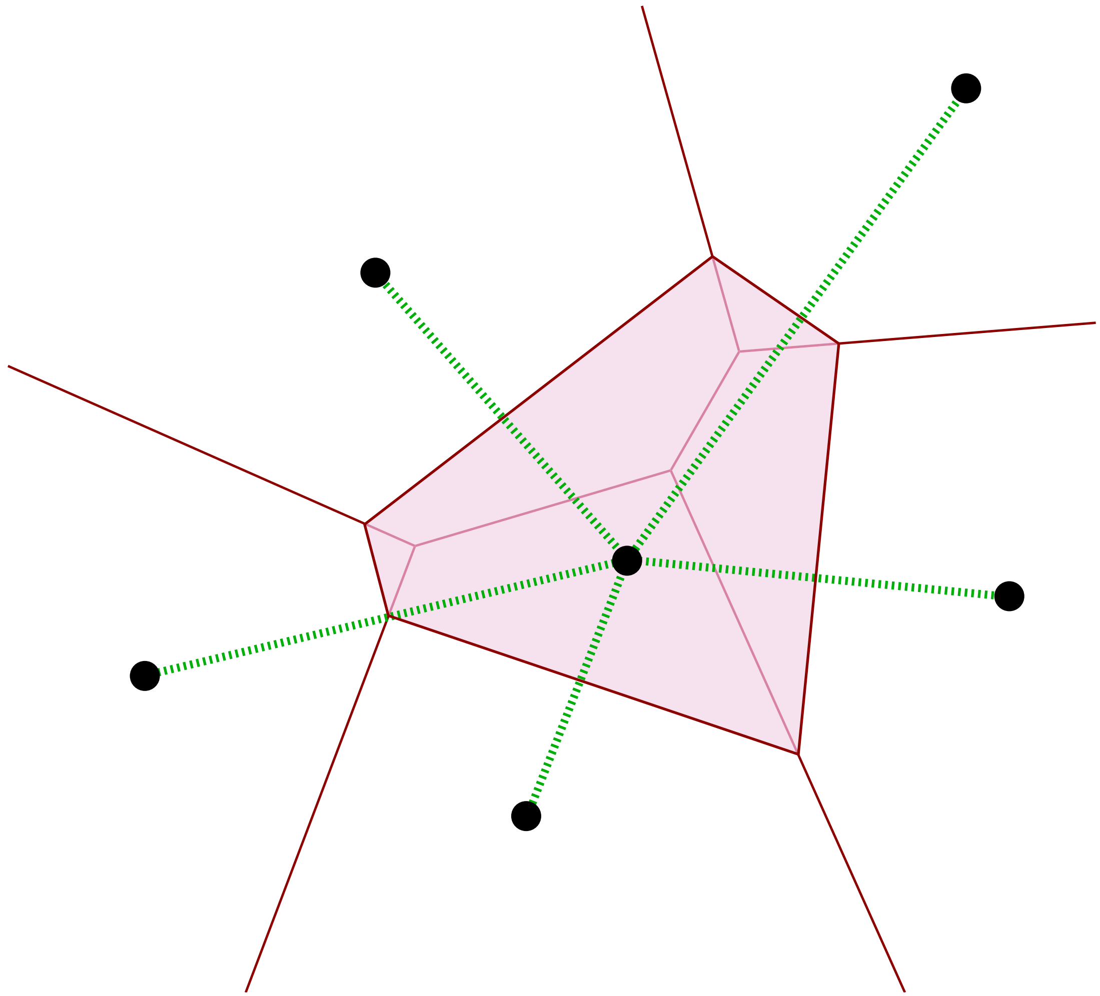


Data may form regular grid or irregular scattered data
here we consider irregular data in two dimensions

Many interpolation algorithms known...

Natural neighbor interpolation [Sibson, 1981]

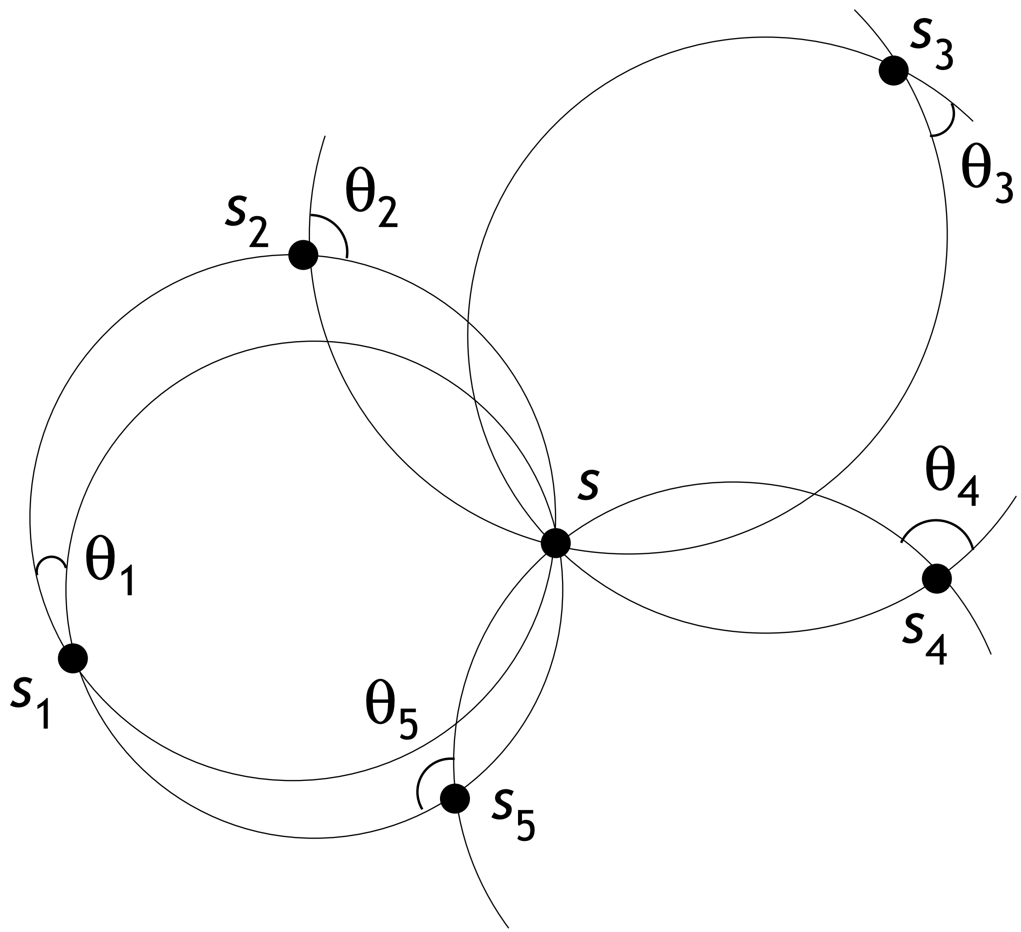
To interpolate at x, add x to Voronoi diagram,
form average of Voronoi neighbors weighted by area
of part of neighbor's Voronoi cell covered by new cell for *x*



Continuous, smooth except at sample points
Correctly reconstructs linear functions

What to use for weights?

Voronoi areas not invariant under Möbius transformation
Instead, use functions of angles between Delaunay circles



Alternative interpretation of angles:
Transform plane so interpolated point goes to infinity
Use angles of convex hull of transformed samples

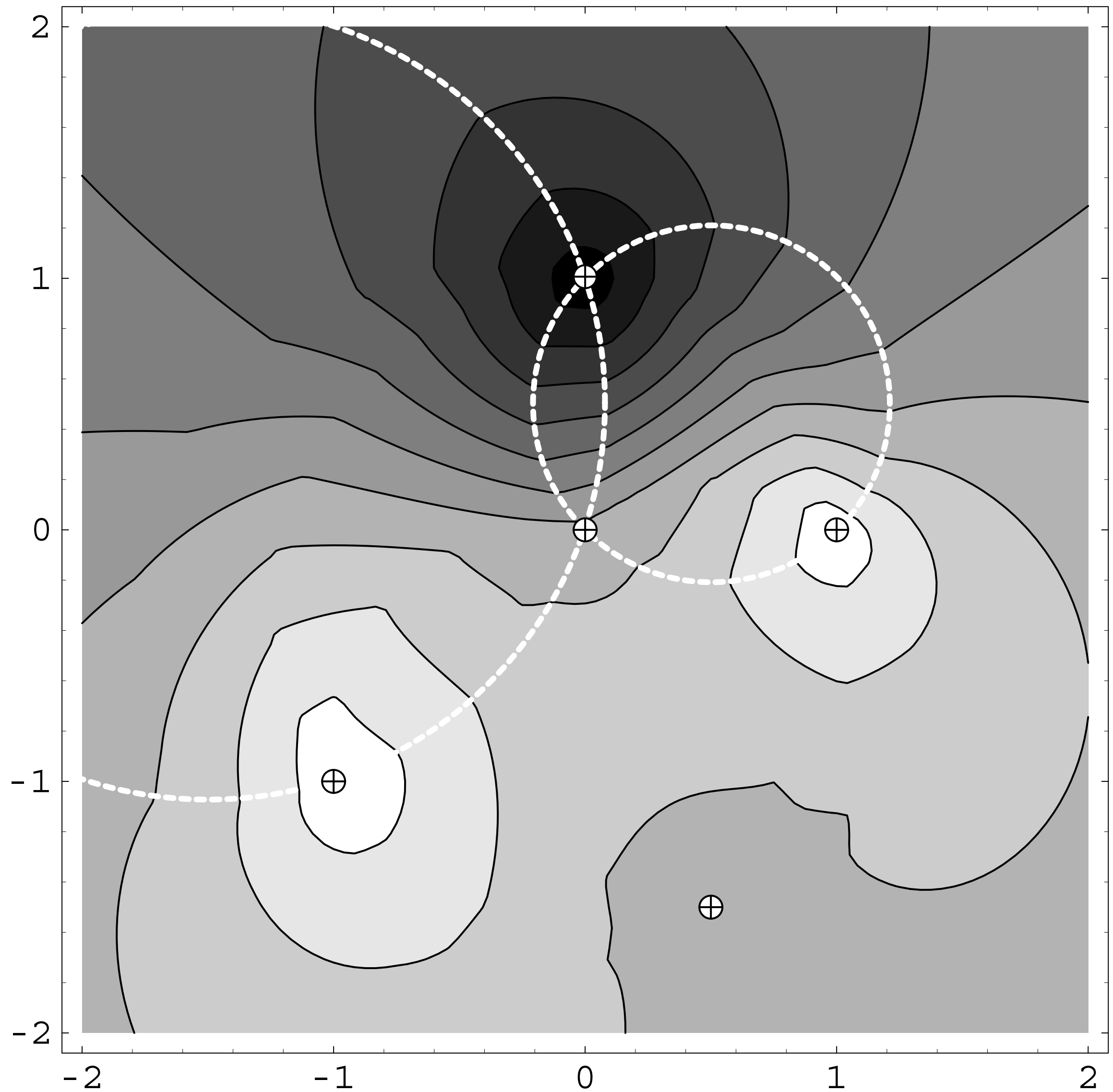
Möbius-invariant interpolation results

Use neighbor-based interpolation
With neighbors = extended natural neighbors
Weights = tan(Delaunay circle angle / 2)

(1) Result is a continuous function
interpolating the sample data

(2) Let f be any harmonic function on a closed disk
and let ε = maximum distance between samples on disk boundary.
Then as $\varepsilon \to 0$, the interpolation converges to f.

Contour plot of example interpolation



Note lack of smoothness along Delaunay circles...

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Initial studies of Möbius transformation in computational geometry

Linear time algorithms for selecting optimal transformation
applications to brain flat mapping, mesh generation

Möbius-invariant natural neighbor transformation
continuous, correctly interpolates sample points
approximately reconstructs of Harmonic functions
applications?

Open Questions

Extend proof of quasiconvexity to more general optimal Möbius transformation problems

Our interpolation is not smooth
Is there a natural choice of smooth Möbius-invariant interpolation?
What about interpolation in higher dimensions?

Other uses of Möbius transformation in computational geometry and scientific computing?