Recognizing Partial Cubes in Quadratic Time

David Eppstein
Computer Science Dept.
Univ. of California, Irvine

Graph theory:
- Unweighted graphs
- Weighted graphs
- Finite metric spaces

Geometry:
- Real vector spaces
- Integer lattices
- Euclidean distances
- \( L_1 \) distances
- \( L_\infty \) distances

Probabilistic tree embedding

Bourgain’s theorem

Johnson-Lindenstrauss lemma …
Context: Geometric graphs and metric embedding

Graph theory:

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- Weighted graphs
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Geometry:

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- $L_1$ distances
- $L_\infty$ distances

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Partial cubes as geometric graphs

Partial cube:

Undirected graph that can be embedded into an integer lattice so that graph distance = $L_1$ distance

At expense of high dimension can restrict coordinates to 0 or 1
$L_1$ distance = Hamming distance: isometric hypercube subgraph

Example: permutahedron
(vertices = permutations of 4 items
edges = flips of adjacent items)
Application: Preference modeling in mathematical behavioral sciences

Given a fixed set of candidates

Model voter states as vertices
Possible state transitions as edges

Several natural families of orderings define partial cubes in this way:

- total orderings
- partial orderings
- weak orderings (total orders with ties)
Application: Modeling knowledge of students

State of knowledge
= set of concepts the student understands

Assume:
Any state can be reached by learning one concept at a time

Union of two states is another state

Then family of states is an antimatroid, a special case of a partial cube

This theory is used by ALEKS Corp. in their educational software for high school mathematics
Application: flip distances in computational geometry

Vertices = triangulations (here, of 3x3 grid)

Edges = change triangulation by one edge ("flip")

Important open problem in algorithms: compute flip distance

Flip graph is a partial cube iff no empty pentagon, polynomial time in this case
For more applications...
Algorithmic problem: efficiently recognize partial cubes

Given as input an undirected graph, produce as output a labeling, and check that the labeling preserves distances

Known: $O(nm)$ time [Aurenhammer and Hagauer, 1995]
Note that $O(nm)$ is $O(n^2 \log n)$ because partial cubes have $O(n \log n)$ edges

Lower bound: output may have $\Omega(n^2)$ bits (e.g. when input is a tree)

New result: $O(n^2)$ time
Graph-theoretic characterization

Djokovic-Winkler relation on graph edges [Djokovic 1973, Winkler 1984]:

\[(p,q) \sim (r,s) \text{ iff } d(p,r) + d(q,s) \neq d(p,s) + d(q,r)\]

\[D + 2\]

G is a partial cube iff it is bipartite and DW-relation is an equivalence relation.

Equivalence classes cut graph into two connected subgraphs.

0-1 lattice embedding: coordinate per class, 0 in one subgraph, 1 in the other, unique up to hypercube symmetries.
Partial cube as finite state machine

Input token \((i,j)\): set \(i\)th bit to \(j\), if possible
otherwise, leave state unchanged

![Diagram of a partial cube as a finite state machine](image)
Automaton-theoretic characterization

Medium [e.g. Falmagne and Ovchinnikov 2002]:

System of states and transformations of states ("tokens")

Every token $\tau$ has a "reverse" $\tau^R$:
for any two states $S \neq V$, $S\tau = V$ iff $V\tau^R = S$

Any two states can be connected by a "concise message":
sequence of at most one from each token-reverse pair

If a sequence of effective tokens returns a state to itself
then its tokens can be matched into token-reverse pairs

States and adjacencies between states
form vertices and edges of a partial cube
Fundamental components of a partial cube

**Vertices** and **edges**, as in any graph, but also:

- equivalence classes of DW-relation ("zones")
- alternatively: tokens or token-reverse pairs
- coordinates of cube embedding
- semicubes (subgraphs cut by equivalence classes)
The Algorithm — overall outline

I. Find a labeling
(distance-preserving iff the input is a partial cube)

Uses Djokovic-Winkler relation

Sped up by bit-parallel programming techniques

II. Check whether it’s distance-preserving

Based on fast all-pairs shortest path algorithm for media

Uses media-theoretic characterization
The Algorithm — finding a labeling

Perform a **breadth first search** from a high-degree root vertex

Label each node by a bitvector
Indicating **which neighbors of root it can connect through**

Label edge by exclusive or of endpoint labels
(should be either zero or single bit)

Sets of edges with same nonzero labels
  = **Djokovic-Winkler classes**

Contract labeled edges and continue in remaining graph
The Algorithm — finding a labeling

Example:
The Algorithm — checking the labeling

Perform a depth-first traversal of the graph, maintaining:

- a list of tokens on shortest paths to the current vertex (one token from each token-reverse pair)
- for each other vertex, the first effective token on the list

When the depth-first traversal moves to another vertex:

- remove the corresponding token from the list, and add its reverse to the end of the list
- for each vertex pointing to the removed token, search forwards for the next effective token

If the search runs off the end of the list, the graph is not a partial cube
The Algorithm — analysis

I. Finding the labeling

Search from degree d vertex finds $d \geq \frac{m}{n}$ tokens using $O(m)$ bitvector operations taking time $O(1 + d/\log n)$ per bitvector operation

Total per token: $O\left(\frac{m}{d} + \frac{m}{\log n}\right) = O(n)$

Whole graph has $O(n)$ tokens, so $O(n^2)$ total

II. Checking whether it’s distance-preserving

Total number of tokens added to end of list: $O(n)$

Each node scans list once, so $O(n^2)$ total
The Algorithm — implementation

220 lines of Python
(approximately 1/3 of which are unit tests)
http://www.ics.uci.edu/~eppstein/PADS/PartialCube.py

Two problematic graphs
(minor bugs in implementation, both fixed, no change to algorithm):

left: crashed the program
right: incorrectly reported as a partial cube