Treetopes And Their Graphs

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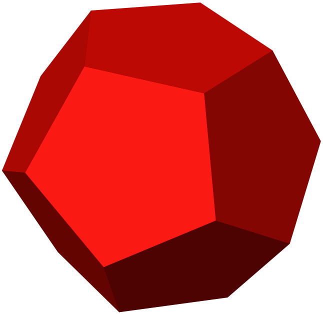
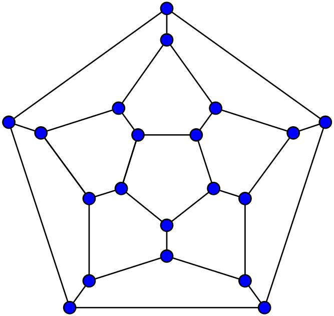
Two possibly NP-intermediate problems, I

Steinitz's theorem:
Graphs of 3d convex polyhedra = 3-vertex-connected planar graphs
[Steinitz 1922]

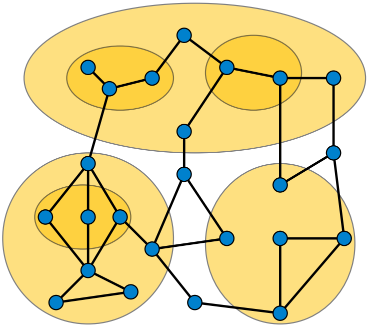




File:Uniform polyhedron-53-t0.svg and File:Graph of 20-fullerene w-nodes.svg from Wikimedia commons

Open: Complexity of recognizing graphs of 4d convex polytopes?

Two possibly NP-intermediate problems, II



Clustered planar drawing:
Visual representation of a graph + hierarchical clustering

Draw graph without crossings

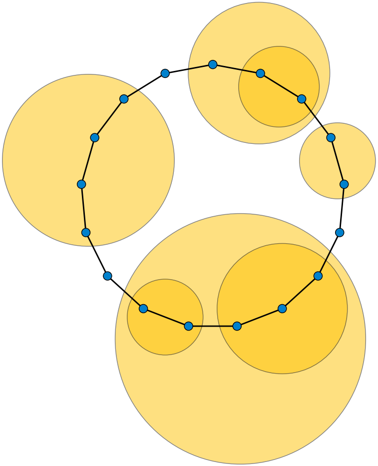
Draw clusters as disjoint Jordan curves

Avoid unnecessary edge-cluster crossings

[Feng et al. 1995; Cortese et al. 2008]

Open: Complexity of finding cluster planar drawings?

A suggestive example



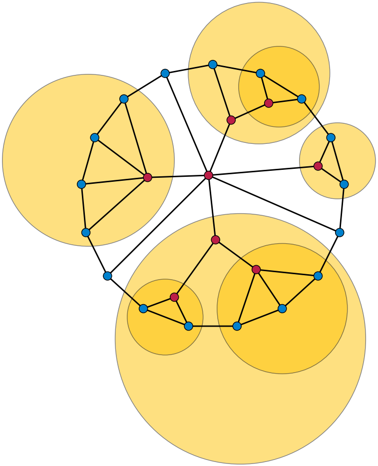
Graph = cycle

Clusters = paths

Can always be drawn as a clustered planar drawing

(Assumptions for later:
≥ 2 vertices/cluster
no complementary clusters)

A suggestive example, II



Add a vertex for each of the regions formed by the Jordan curves

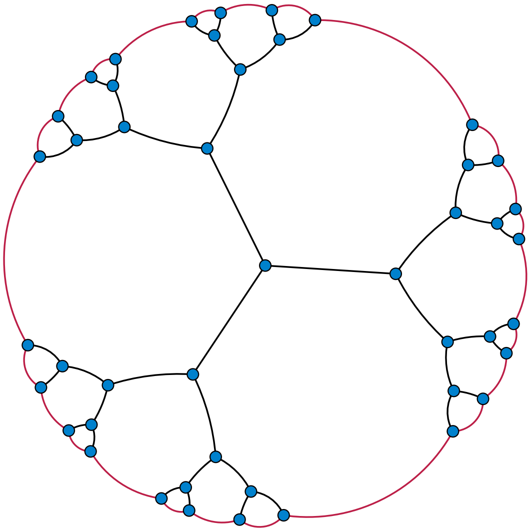
Connect region vertices to the graph vertices in their region

Connect vertices for adjacent pairs of regions

The result is a Halin graph!

...and all Halin graphs can be formed in this way

Halin graphs



Draw a tree in the plane

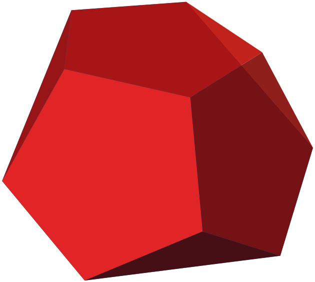
- No crossings
- No degree-two vertices

Connect the leaves by a cycle that contains the tree

Halin graph history

Studied by [Halin 1971] as a class of minimally 3-connected graphs:
3-connected, but removing any edge or vertex breaks this property

⇒ meet conditions of Steinitz's theorem, form polyhedra



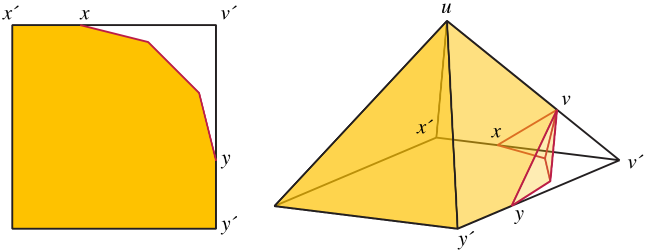
Also known as "roofless polyhedra" or "based polyhedra"
[Kirkman 1856; Rademacher 1965]

Halin graph polyhedral realization

Find a tree vertex *v* all of whose children are leaves

Remove *v*'s children and realize the smaller graph by induction

Move *v* towards its parent on the edge connecting them, replacing it in the base face with a convex chain formed by its children



Used by [Aichholzer et al. 2012] to find realizations with horizontal base, all other faces having equal slopes (realizing any tree as a medial axis or straight skeleton)

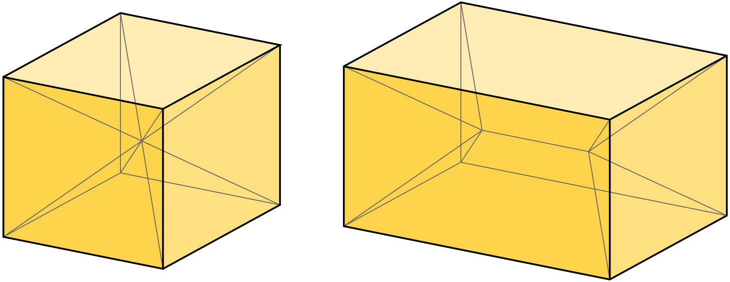
The question that started this line of research

What is the right high-dimensional generalization of a Halin graph?

Answering this leads to results that connect both 4-polytope recognition and clustered planarity

Treetopes

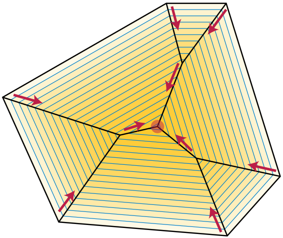
Polytopes with a base facet (outermost in these Schegel diagrams)
s.t. each face of dim ≥ 2 shares ≥ 2 vertices with the base



Pyramid over cube (left) and prism over square pyramid (right)

Why "treetopes"?

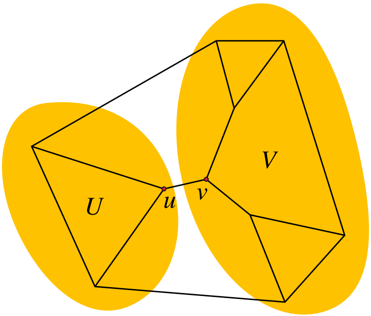
Edges not in base form a tree, the "canopy" of the treetope



Proof sketch (in any dimension):

- Transform so that no two non-base vertices have equal distance from base plane
- Use simplex method to maximize distance from base ⇒ tree leading to farthest vertex
- Never more than one distance-increasing edge from any vertex, because then that vertex would be the bottom vertex of a face disjoint from the base

Graph clusterings from treetopes

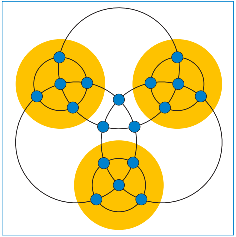


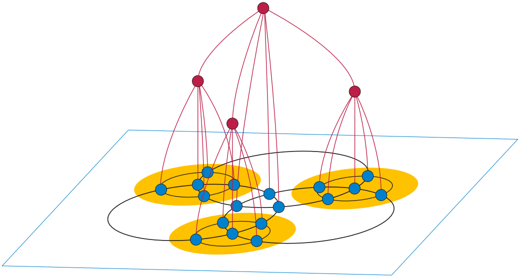
Removing any canopy edge *uv* partitions canopy into two subtrees $\Rightarrow$ clusters of base

Any two non-complementary clusters share at most one edge

Contracting any cluster preserves base graph $(d-1)$-connectivity

Cluster graph from clustering

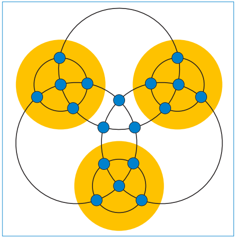


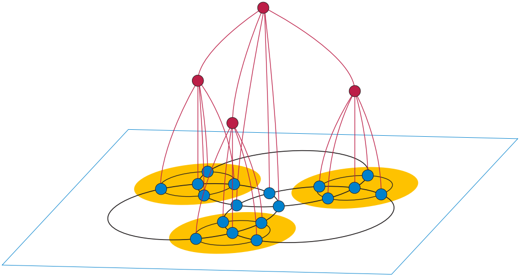


Instead of drawing the clustering and using regions:

- Keep only one cluster for each complementary pair (so each two clusters are disjoint or one is a subcluster of the other)
- Add cluster of all vertices
- Create new vertex for each cluster, adjacent to its maximal subclusters and unclustered vertices

Treetopes from clustered planar graphs





Theorem: Treetope = cluster graph of a clustering such that:

Underlying planar graph (base facet of treetope) is 3-connected

Collapsing any cluster or complement gives a 3-connected minor

Each cluster vertex has degree at least four

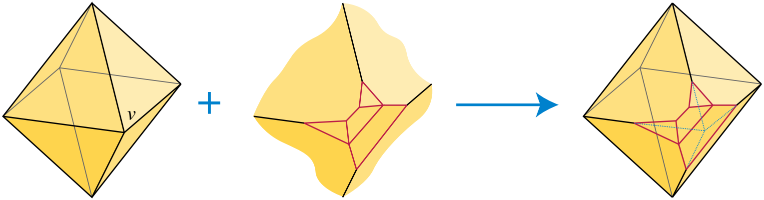
At most one edge connects each two disjoint clusters,
complements or single vertices, unless they cover whole graph

Proof idea: Inductive realization of treetopes

Same basic idea as Halin graph realization

Induction on # clusters:

Collapse a cluster to a vertex, realize inductively, uncollapse



Uncollapse = replace base polyhedron vertex by polyhedral surface

To find the surface, use (polar version of) a result that any 3d polyhedron can be realized with one face shape specified

[Barnette and Grünbaum 1970]

Polynomial time recognition of 4-treetopes

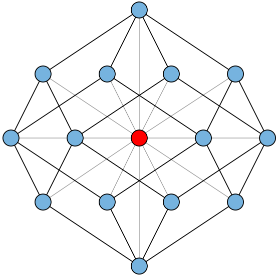
Basic idea:

Recognize a minimal cluster

Collapse cluster into a single vertex

Repeat until stuck (not a treetope)

or we reach a pyramid over a polyhedral graph (success)



Problem: some base vertices look like cluster vertices

Solution: they still lead to valid cluster collapses

Polynomial time recognition, details

Repeat:

- Find a vertex v that looks like a cluster vertex
  - At least four neighbors
  - Its neighborhood = planar graph + isolated vertex
  - No two neighbors adjacent to same non-neighbor
  - Delete it and contract non-neighbors ⇒ 3-connected
  - Not marked as part of base polyhedron
- Contract v and its neighbors
- Mark contracted vertex as part of base

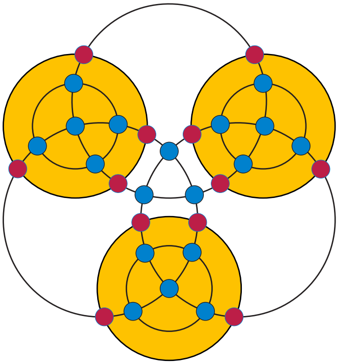
Verify that this process reduces to polyhedron + universal vertex

Additional properties: Separator theorem

Graphs of 4-treetopes can be bisected by removal of $O(\sqrt{n})$ vertices

Proof idea:

- Construct clustered planar drawing
- Replace cluster boundaries by edge cycles, crossings by vertices
- Use planar graph separator theorem



False for simple (4-regular) 4-polytopes [Loiskekoski and Ziegler 2015]

For more general clustered planar drawings, planarization can be nonlinear; do their cluster graphs have good separators?

Conclusions

New class of polytopes, defined in all dimensions,
generalizing the Halin graphs

Four-dimensonal case can be recognized in polynomial time,
has useful algorithmic properties such as small separators

Open: can 4-treetopes be realized in polynomial time?

Subproblem: can 3d polyhedra with a specified face shape be
realized in polynomial time?

Bigger open problems:

Complexity of recognizing graphs of arbitrary 4-polyhedra

Complexity of recognizing clustered planar graphs

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