Shortest Paths in an Arrangement with k Line Orientations

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Abstract

Suppose one has a line arrangement in which many lines are parallel, so that the number of different line orientations is k, and one wants to find a shortest path from one point on a line in the arrangement to another such point. Using known techniques one can find the shortest path in time and space $O(n^2)$. We present an algorithm that can find the shortest path in time and space $O(n + k^2)$.

1 Introduction

Given a line arrangement and two points located on lines of the arrangement, it is natural to ask how to compute a shortest path from one point to the other, traveling only on lines of the arrangement. Of course, one can simply construct the arrangement, interpret its vertices and edges as forming a planar graph, and find a shortest path in this graph. It is well known how to compute arrangements in time $O(n^2)$, and the resulting planar graph has $O(n^2)$ vertices and edges. Klein et al. [4] show how to compute shortest paths in planar graphs in linear time, hence the shortest path in the arrangement can be found in time and space $O(n^2)$.

For some time, it has been an open problem to improve this bound [3, 5]. In a recent paper, Bose et al. [1] made some progress on this question by describing an $O(n \log n)$-time algorithm that finds a path approximating the length of the shortest path to within a factor of two.

In previous work [2], we looked at a related shortest path problem: given a set of vertical and horizontal line segments, how can one quickly compute a shortest path from one point in the arrangement to another? We showed an $O(n^{3/2} \log n)$ bound for this problem; subsequently, Marc van Kreveld (personal communication) pointed out that $O(n \log n)$ is achievable by applying a rectilinear shortest paths algorithm to a set of obstacles derived from the nonconvex cells of the arrangement. However, these methods do not seem to generalize to arrangements with more than two orientations.

In this paper, we consider arrangements with few distinct line orientations.[3] We show that only a constant number of lines of each orientation can participate in the shortest path, and that these lines can be identified quickly. Hence, if there are k orientations, the shortest path can be found in time and space $O(n+k^2)$. If the n lines have n different orientations, this time bound becomes $O(n^2)$. Clearly, the result we present is the best one can do without finding a faster way to solve the shortest path problem in an arrangement with unrestricted line orientations.

Also interesting is the fact that the key lemma for obtaining this result *does not use the limit on line orientations; it applies to any arrangement.* For an unrestricted arrangement, one can use this lemma to remove lines that don't contribute to a shortest path. After applying this lemma, one is left with lines that form two (somewhat irregular) checkerboard patterns connected at the corners. One would expect that finding a shortest path in this simplified arrangement would be much easier, but in fact the problem is still difficult, and improving on the $O(n^2)$ time remains an open problem. (One can now, however, find the length of the shortest path in $O(n)$ space.)

2 The Algorithm

Our shortest paths algorithm is very simple, as is its time and space analysis. It is only its proof of correctness that is complicated. Given an arrangement of n lines, and endpoints s and t on lines of the arrangement (we assume s and t are on the x axis), we perform the following steps:

1. For each of the k orientations, find the nearest line to the left of s, the nearest line to the right of s, the nearest line to the left of t, and the nearest line to the right of t.

2. Construct the arrangement of these at most $4k$ lines.

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[3] We assume that the lines are partitioned by orientation. If not, they can be sorted in $O(n \log k)$ additional time.

3. Apply the shortest path algorithm of Klein et al. [4] to the resulting planar graph.

Theorem 2.1. *The algorithm described above computes the shortest path from s to t in time and space $O(n + k^2)$.*

We will spend the rest of the paper proving the theorem by showing that the $4k$ lines selected by the algorithm are sufficient to find a shortest path.

3 Interior and Exterior Lines

Throughout, we let A denote an arrangement of n lines $\ell_1, \ell_2, \ldots, \ell_n$. We let s and t denote the two points between which we wish to compute the shortest path, where s and t lie on lines of the arrangement. Without loss of generality, we assume that s and t lie on the x axis of the plane, with s to the left of t (i.e., having smaller x-coordinate than t).

Define an *interior line* to be a line of the arrangement which passes between s and t (to the right of s and left of t) and an *exterior line* to be any line of the arrangement that is not interior (both s and t are in the same half plane bounded by the line). We also define a line to be exterior if it passes through s or t. Define an *exterior halfspace* to be the halfspace bounded by an exterior line and containing both s and t.

We will repeatedly use the fact that intersection of a shortest path with any line of the arrangement, if non-empty, is either a point or a contiguous line segment. This follows easily from the facts that a line is the shortest distance between two points, and that any contiguous portion of a shortest path must itself be a shortest path between its endpoints. The first consequence of these facts is the following:

Lemma 3.1. *The shortest path from s to t is contained in the intersection of the exterior halfspaces.*

Proof. The path cannot cross any exterior line since it would then have two disconnected intersection points on that line.

As a corollary, we can eliminate all exterior lines except for the at most $2k$ that contribute segments to this intersection. This set is a subset of the lines selected by our algorithm.

4 Unimodal Sequence of Angles

Define an *interior path* to be a contiguous portion of the shortest path from s to t, lying only on interior lines.

It is natural to conjecture that the shortest path uses at most one interior line, that it only uses one interior path, or that each interior path involves at most one line. If true, such facts would greatly simplify the problem of finding shortest paths. However, they are not true; see Figure 1. As the figure shows, it is also not true that the shortest path will use at most one line of each orientation, or that multiple parallel lines used by the shortest path must be traversed in a consistent direction.

Fortunately, the interior paths do have some important structural properties. Recall that we are assuming x and y are on the x axis; note that no interior line can be horizontal. It is helpful to give an orientation to the path: we will say that the path travels from s to t. Then we say that a line *travels up* if the shortest path follows that line along a segment in which the y coordinate increases as one progresses from s to t, and that the line *travels down* otherwise. Note that it is not possible for a line to be both up and down since the shortest path can only have one segment on the line.

Lemma 4.1. *If a line in an interior path travels up, the next line travels down, and vice versa.*

Proof. Let the first line be ℓ_i and the second be ℓ_j, and assume that line ℓ_i travels up. Let x denote the point where they intersect, so the portion of ℓ_i belonging to the shortest path is below x. The portion of ℓ_i below x can not be to the right of ℓ_j, because if it were we would have crossed ℓ_j to reach ℓ_i and the shortest path cannot cross a line twice. But a symmetric argument shows that if ℓ_j travels up, the portion of ℓ_j above x cannot be to the left of ℓ_i. However we assumed the two lines cross at x, so ℓ_j must travel down.

If ℓ_i travels down a symmetric argument shows that ℓ_j travels up.

Lemma 4.2. *Suppose we have a shortest s-t path that, among all shortest paths, minimizes the number of vertices, with four consecutive lines ℓ_g, ℓ_h, ℓ_i, and ℓ_j on the path meeting at points p_{gh}, p_{hi}, and p_{ij}. Suppose also that ℓ_h and ℓ_i are interior lines. Then the angle formed by the path at p_{hi} is not the sharpest of these three angles.*

Proof. Because ℓ_h and ℓ_i are interior, and by Lemma 4.1, the angles on the path must alternate in orientation. If the middle one were the sharpest, then the quadrilateral formed by these four lines would be completely contained in the parallelogram formed by ℓ_h, ℓ_i, and lines parallel to these two lines through points p_{gh} and p_{ij} (Figure 2). It does not equal the parallelogram since if it did, the path directly from ℓ_g to ℓ_j would have equal length and fewer vertices. But then the path from p_{gh} to p_{ij} around the other side of the parallelogram from p_{hi} has equal length to the given

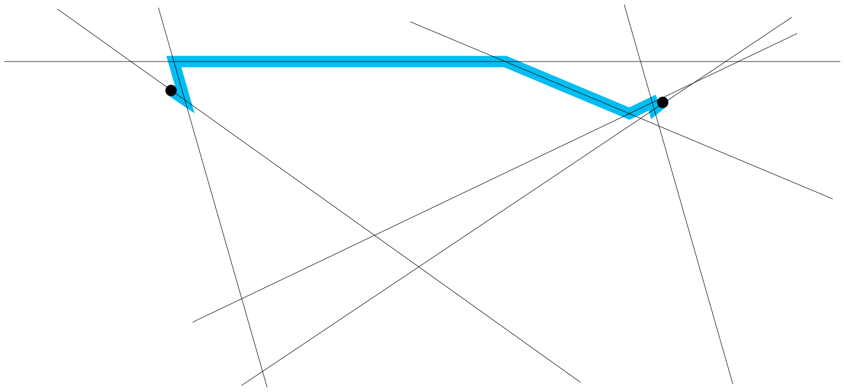

Figure 1. Shortest paths may use multiple interior paths, and multiple interior lines per path. The distance to the destination may not be monotonically decreasing along the path. The path may traverse parallel lines in opposite directions.

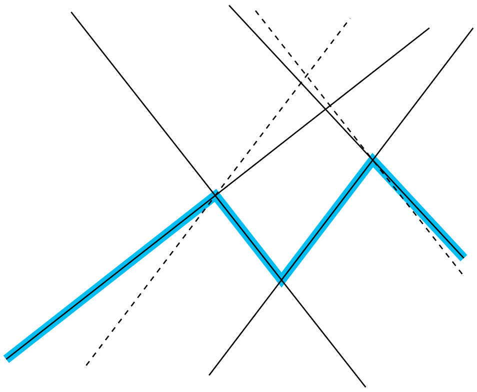

Figure 2. A sharp angle between two shallower angles forms a quadrilateral completely contained in a parallelogram, showing that the original path cannot be shortest.

path, and can be shortened by instead going around the other side of the quadrilateral. (The parallelogram path can be transformed to the quadrilateral path by two steps each of which replaces two sides of a triangle by the third side.) This contradicts the assumption that the original path was shortest.

Corollary 4.1. *The angles on an interior path form a unimodal sequence, which is strictly increasing to a maximum (the maximum may repeat) and then is strictly decreasing.*

Note that unimodality applies as well to the angle preceding the interior path (between an exterior line and the first interior line) and following it.

5 The Better Arrangement Lemma

In order to use the unimodality property shown above, we consider changing the arrangement and discovering how these changes affect its path lengths.

Lemma 5.1. *Consider three lines containing three consecutive segments of a shortest path. Assume that the first and third lines intersect at a point p, and assume that travel on the first and third lines is in the same direction (both toward p or both away from p). If one translates the middle line toward p without crossing a vertex of the path, then the path on the translated line (traveling on the same sequence of lines) is shorter.*

Proof. Let ℓ_1, ℓ_2, and ℓ_3 denote the three lines. Consider an imaginary line that passes through the intersection point of of ℓ_1 and ℓ_2 and is parallel to ℓ_3. We have a copy of ℓ_2 translated to a location closer to p as another line of the arrangement. These lines form a parallelogram (Figure 3). A simple application of the triangle inequality at the imaginary line completes the proof.

Lemma 5.2. *Let ℓ be a line on an interior path consisting of two or more interior lines; let x be the*

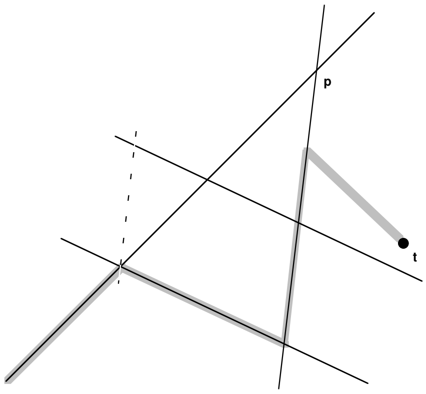

Figure 3. With two parallel lines, the path on the line closer to p is shorter, as is obvious by applying the triangle inequality at the dashed line.

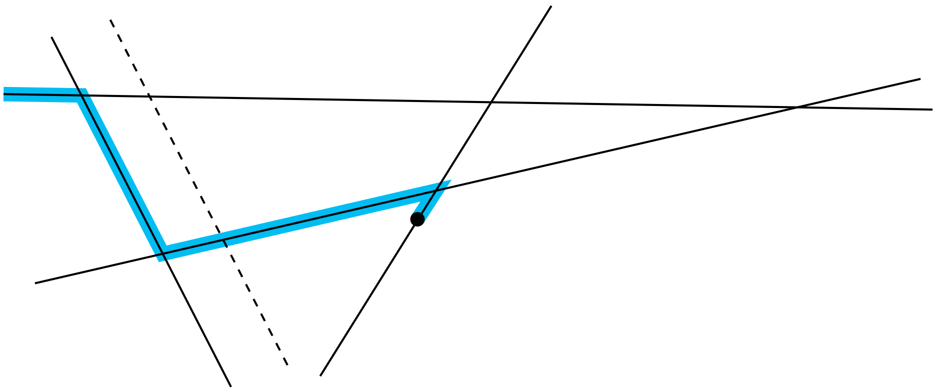

Figure 4. Translating a line away from the shallowest angle produces a shorter path. (The dashed line indicates the new position of the translated line.)

shallowest angle on the interior portion of the path; and let arrangement A' be formed by translating ℓ away from x until it first crosses s, t, or another vertex of the path. Then the path in A' formed by using the same sequence of edges has a length less than or equal to the length of the path in A.

Proof. Let ℓ_i and ℓ_j denote the lines preceding and succeeding ℓ on the path. Assume without loss of generality that ℓ is to the right of x. (This implies that ℓ_i is also an interior line.) If ℓ_i and ℓ_j are parallel, the length of the path in A' is equal to the length in A. Otherwise, as in the proof of Lemma 4.2, the angles alternate in orientation, and thus travel on both ℓ_i and ℓ_j is toward their point of intersection. From the unimodality of the angle sequence, the intersection of ℓ_i and ℓ_j must be to the right of ℓ (Figure 4). The result then follows from Lemma 5.1.

Next we define a measurement of the "quality" of interior lines. Given a set of interior lines, thought of as all upward lines or all downward lines, we say that a line has *best slope* if the angle (a positive number from 0 to π) between it and the x-axis is smaller than the angle of any of the other lines.

Lemma 5.3. *Given three consecutive lines traversed by a shortest path (with the assumptions of Lemma 5.1) where the middle line is an interior line, if there is an interior line closer to the intersection of the first and third lines than the middle line, and that line has better slope, then a path traveling on that line is shorter than the original path.*

Proof. First translate the middle line closer to the intersection, until it intersects the better slope line. The translated line gives a shorter path, and since it intersects the better slope line, two applications of the triangle inequality show a path on the better slope line is shorter still.

6 The Sandwich Lemma

Lemma 6.1. *If a portion of an interior line ℓ is sandwiched between two interior lines ℓ_s and ℓ_t that are both either parallel to ℓ or have better slope than ℓ, then the shortest path between s and t need not travel on this portion of ℓ.*

Proof. Assume that we have a shortest path traveling on this portion of ℓ. We will show that an equal or shorter path must exist that does not use this portion of ℓ. In the case that we find a shorter path, we contradict the assumption that ℓ is on a shortest path; otherwise we have found an equally good path that does not use ℓ.

Assume without loss of generality that ℓ travels up. We know that the path crosses ℓ_s and ℓ_t; let p_s be the point where the path leaves ℓ_s toward ℓ and p_t be the first point where the path contacts ℓ_t.

We look at three cases, depending on where the shortest path travels on exterior lines.

Case 1. The line ℓ_1 immediately preceding ℓ and line ℓ_3 immediately following ℓ are exterior lines.

Because ℓ travels up, the intersection of ℓ_1 with ℓ is below the intersection of ℓ_3 with ℓ. This must also be true of the intersections with ℓ_s and the intersections with ℓ_t as follows. If it were not, then ℓ_1 crosses ℓ_3 in the region between ℓ_s and ℓ_t, and the intersecting lines define four quadrants. Assuming without loss of generality that the intersection is on the left side of ℓ, the one or two intersections with ℓ_s (one or both of ℓ_1 and ℓ_3 must intersect ℓ_s) are in the quadrant opposite the two intersections with ℓ. But the path must cross line ℓ_s in the first of these quadrants, and by assumption it travels on ℓ in the second of the quadrants, contradicting the fact that when two exterior lines intersect, they partition the plane into four regions of which only one can contain any part of the shortest path.

From this, we can conclude that the exterior lines (if not parallel) intersect to the left of ℓ_s or to the right of ℓ_t.

Suppose the exterior lines are parallel. The path departs ℓ_s at some point p_s. The line of departure may be ℓ_1; if not, p_s is above ℓ_1 and below ℓ_3. Consider the (possibly imaginary) line ℓ_i through p_s and through the intersection point of ℓ_1 and ℓ (see figure 5). Line ℓ_i is either parallel to ℓ_3, or it intersects ℓ_3 at a point p to the left of ℓ_s. A path traveling from p_s on ℓ_i to ℓ is as good or better than our shortest path. But, line ℓ_s is closer to p and has as good or better slope than ℓ, so by Lemma 5.3, a path from p_s on ℓ_s to ℓ_3 is equal or better than our shortest path; it does not travel on ℓ; and it exists in the arrangement.

If ℓ_1 and ℓ_3 are not parallel, we assume without loss of generality that they intersect at a point p to the left of ℓ_s. We again look at a (possibly imaginary) line through p_s, and following the same reasoning as above, we have that the path from p_s on ℓ_s to ℓ_3 is shorter than the claimed shortest path.

Cases 2 and 3. At least one line immediately preceding or following ℓ is an interior line.

Let x denote the shallowest angle on the interior path containing ℓ. Line ℓ must be to the left or right of x; assume without loss of generality that ℓ is to the right of x and ℓ_t is the sandwiching line to the right of ℓ. Other cases where travel on ℓ is down or ℓ is to the left of x are completely symmetric.

Case 2. The path from ℓ to the point it reaches ℓ_t does not intersect any external line.

Let b denote the bottom point of the intersection of ℓ with the path.

We repeatedly apply Lemma 5.2 to move line ℓ away from x (and toward t), modifying the arrangement and maintaining as we do a path in the modified arrangement no longer than the original s-t path.

In this process, we may want to move ℓ across a vertex v on the path. By unimodality, that vertex must be the top of an upward-going line segment and must be above b (Figure 6). If this happens, we replace the portion of the path between ℓ and v with a path traveling up ℓ to reach v; since this bypasses two segments of the path (the downward segment immediately after ℓ and the upward segment that reaches v), this new path is shorter.

If one asks what happens in this process if the segment on ℓ shrinks to zero length, the process terminates with a path not involving ℓ. Otherwise the process terminates when the point b at the bottom of the segment ℓ contacts ℓ_t.

Since ℓ_t separates b from t, since ℓ_t has slope as good or better than ℓ, and since the bottom point b on the intersection of the path with ℓ moves continuously to the right, we must eventually reach a point at which that bottom point lies on ℓ_t. But then, the path travels up from that point along ℓ, and must cross ℓ_t to reach t, so it can be replaced by a shorter or equal path using ℓ_t that does not use ℓ. This new path exists in the unmodified arrangement.

Case 3. The path from ℓ intersects an exterior line before it reaches ℓ_t.

We translate ℓ just as in the argument above, moving past vertices where interior lines intersect other interior lines.

Eventually, ℓ comes in contact with a vertex v at the intersection of an interior line ℓ_i of the path and an

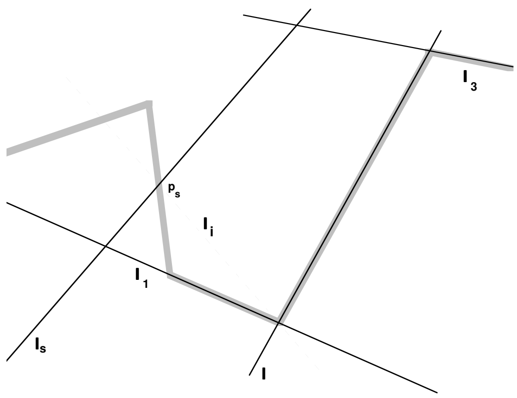

Figure 5. Before reaching line ℓ, the path must cross line ℓ_s at some point, denoted here by p_s. We draw an imaginary line through this point and the point where the path reaches ℓ.

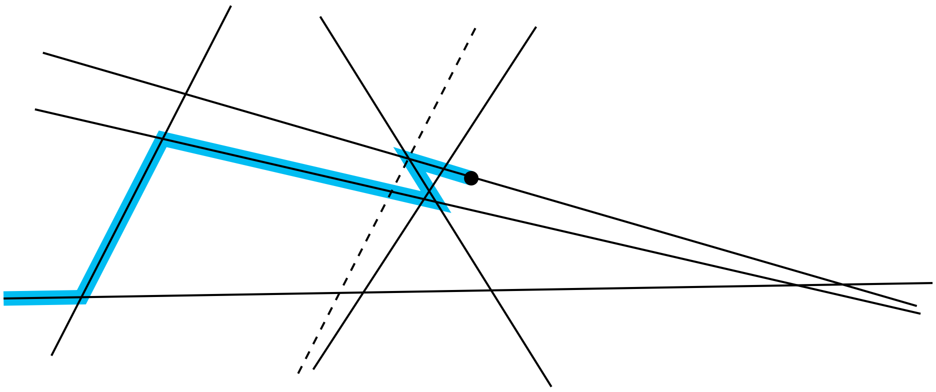

Figure 6. Moving line crosses the path at the top vertex of an upward-going segment.

exterior line ℓ_e of the path.

Suppose that ℓ_i is a downward line. The line prior to ℓ_i is an interior line, so unimodality applies to the angle between ℓ_i and ℓ_e. Thus, travel on ℓ_e is up and ℓ_e makes a greater angle to the x-axis than does ℓ.

By assumption for Case 3, point b on the translated ℓ has not reached ℓ_t, so the portion of the translated ℓ above b is left of ℓ_t, and the portion of ℓ_e above v is left of the translated ℓ; we conclude ℓ_e passes to the left of t. But the path travels down interior line ℓ_i and must turn left onto ℓ_e, so the part of ℓ_e above this point is to the right of ℓ_i, which in turn is to the right of s, contradicting the assumption that ℓ_e is an exterior line.

It must be, then, that ℓ_i is an upward line. Unimodality again applies to the angle at ℓ_i and ℓ_e.

A shortest path must cross ℓ_t at some point p_t, which must be below (or on) ℓ_e and above ℓ_1. Just as in Case 1, we look at a (possibly imaginary) line through p_t and vertex v. The same argument as in Case 1 gives the conclusion that a path traveling on ℓ_t, instead of ℓ, is equal or shorter than the claimed shortest path.

7 Correctness

We now conclude the proof of Theorem 2.1.

Lemma 7.1. *If we have any three parallel interior lines, the middle one of the three need not appear on a shortest path.*

Proof. This is just a special case of Lemma 6.1.

Lemma 7.2. *If we have a class of parallel lines, the only ones that may need to be included in a shortest path are those nearest to each of s and t on their left and right sides.*

Proof. The lines to the right of s and left of t are interior, and by Lemma 7.1 we need only consider the outermost of these (we can use the lemma to remove the remaining lines one at a time from the arrangement without changing the shortest path length). The lines to the left of s are exterior, and by Lemma 3.1 we need only consider the rightmost of these; similarly we need only consider the leftmost of the lines to the right of t.

Corollary 7.1. *The algorithm described in Theorem 2.1 is correct.*

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