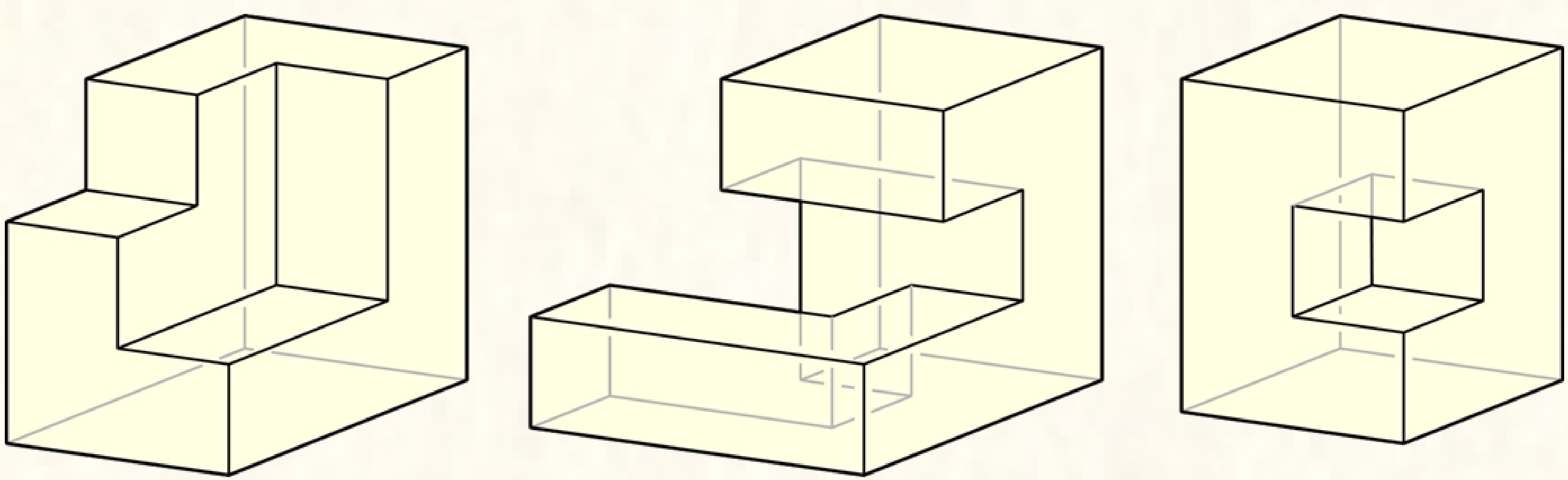
Steinitz Theorems for Orthogonal Polyhedra

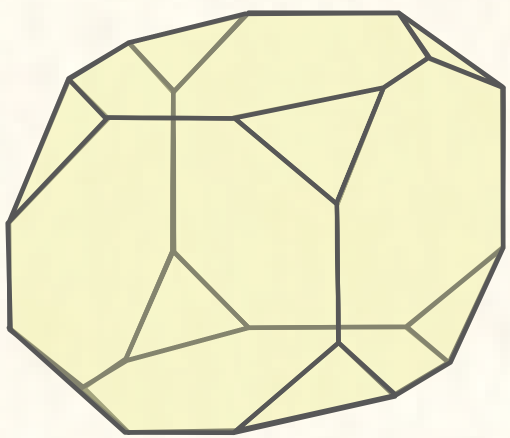
David Eppstein
and
Elena Mumford

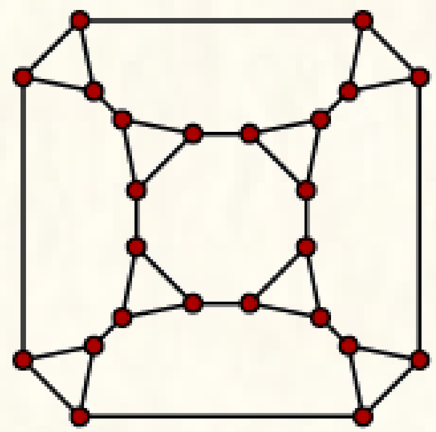


Steinitz Theorem for Convex Polyhedra

Steinitz:

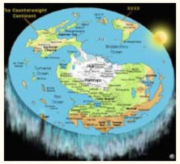
skeletons of convex polyhedra in $\mathbb{R}^3$ = planar 3-vertex-connected graphs



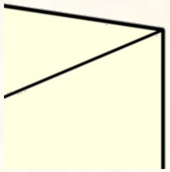


Simple Orthogonal Polyhedra

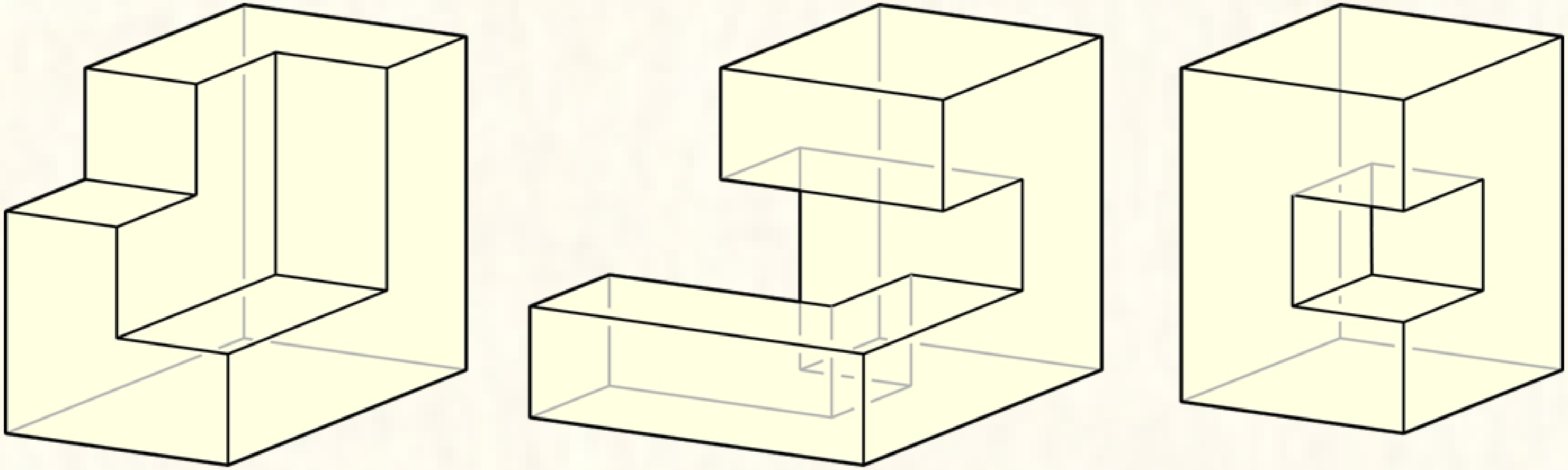
Topology of a sphere



Simply connected faces



Three mutually perpendicular edges at every vertex



simple orthogonal polyhedra

Simple Orthogonal Polyhedra

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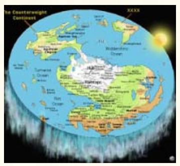
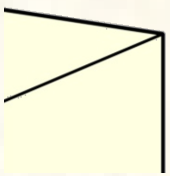


Simply connected faces



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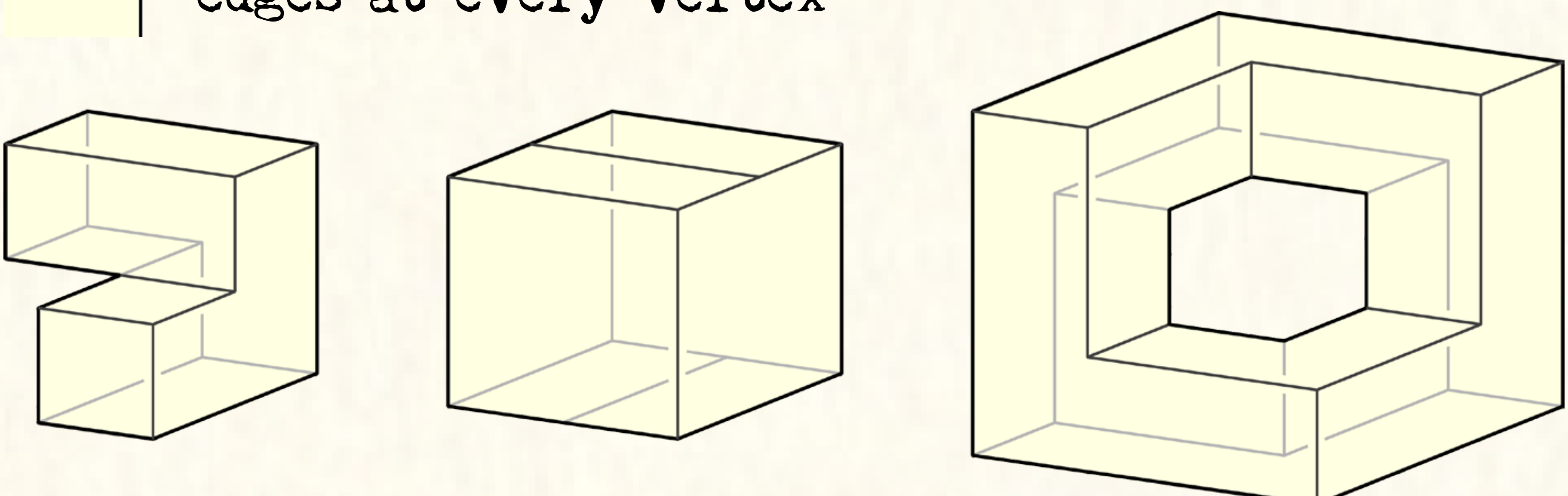
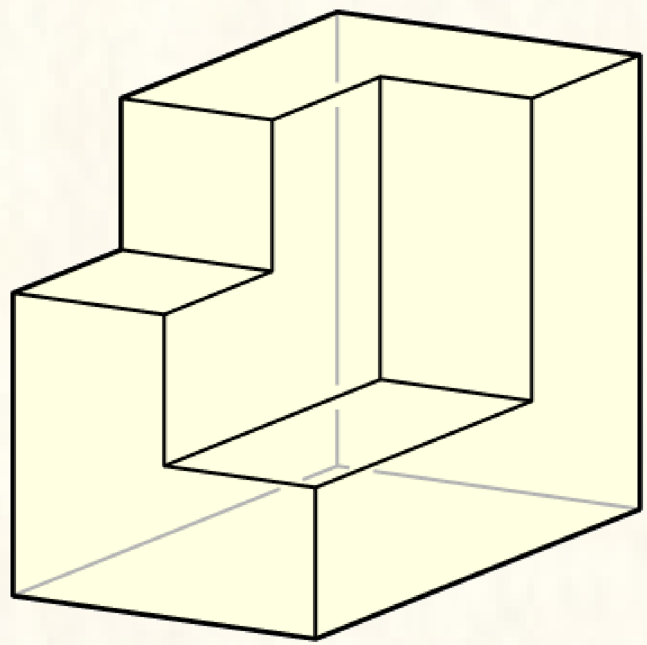


Orthogonal polyhedra that are NOT simple

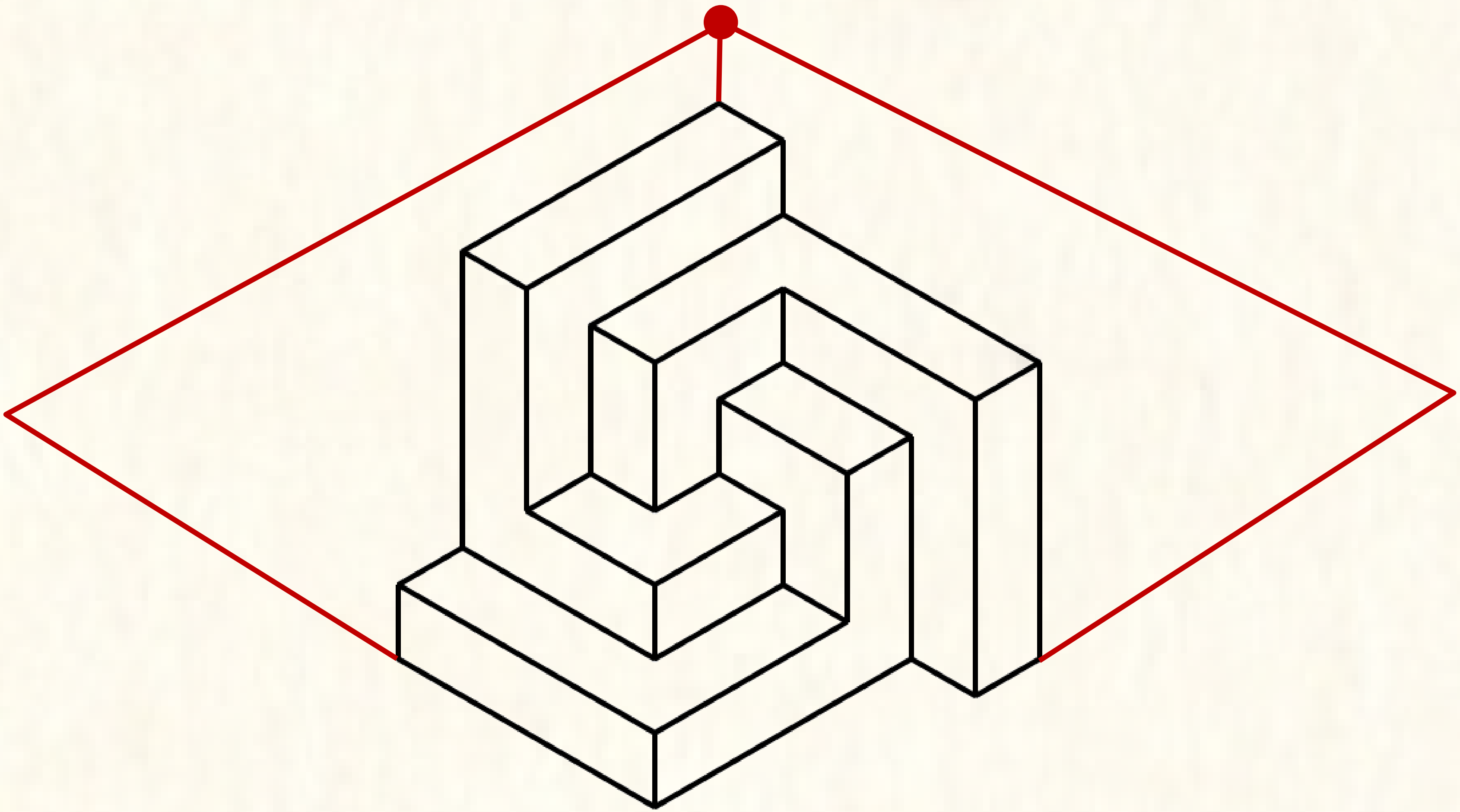
Corner polyhedra

All but 3 faces are oriented towards vector (1,1,1)



= Only three faces are "hidden"

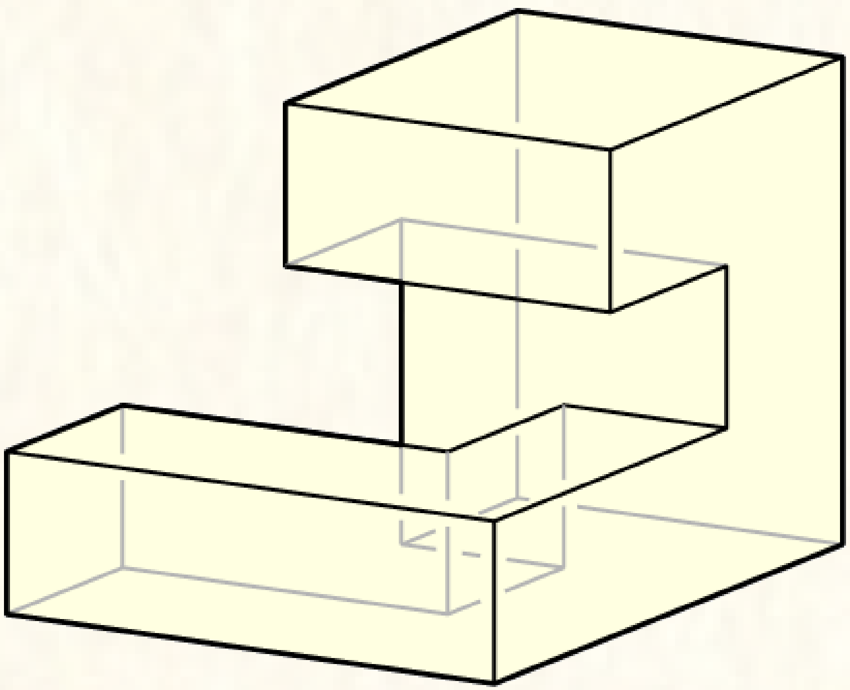
Corner polyhedra

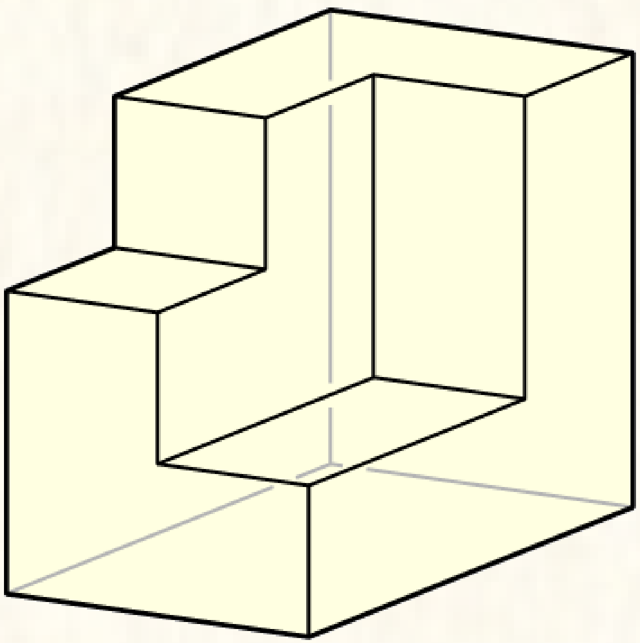


Hexagonal grid drawings with two bends in total

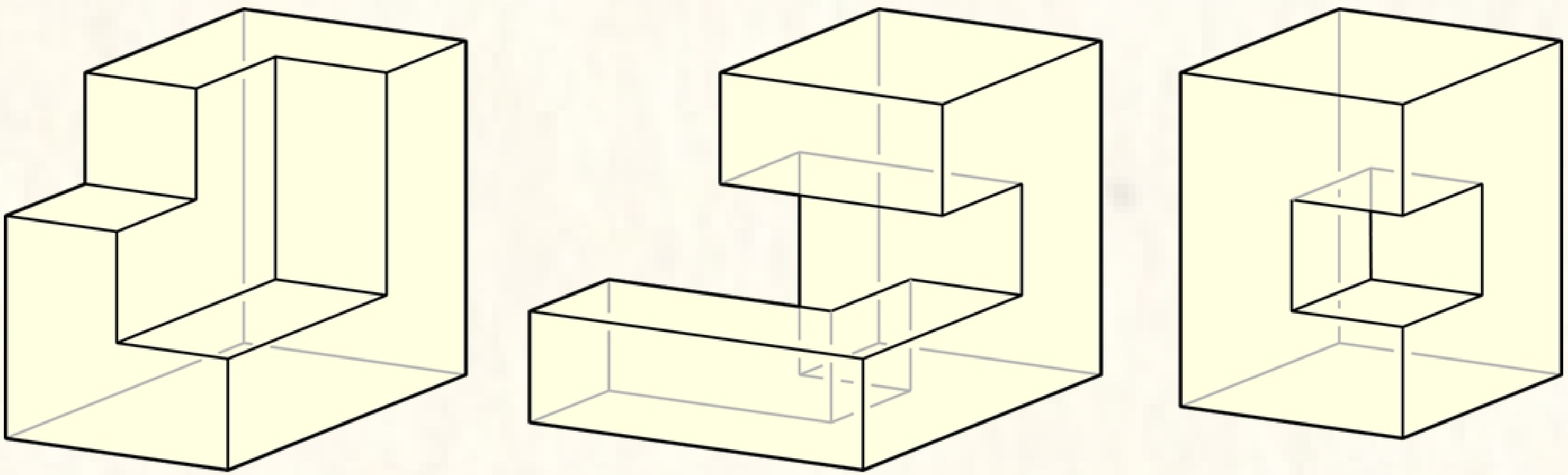
XYZ polyhedra

Any axis parallel line contains at most two vertices of the polyhedron





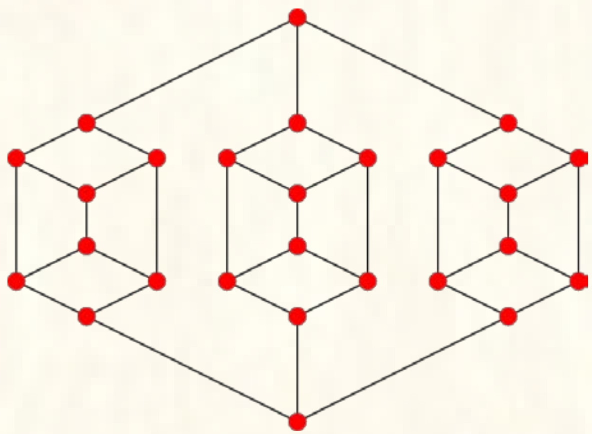
Skeletons of Simple Orthogonal Polyhedra



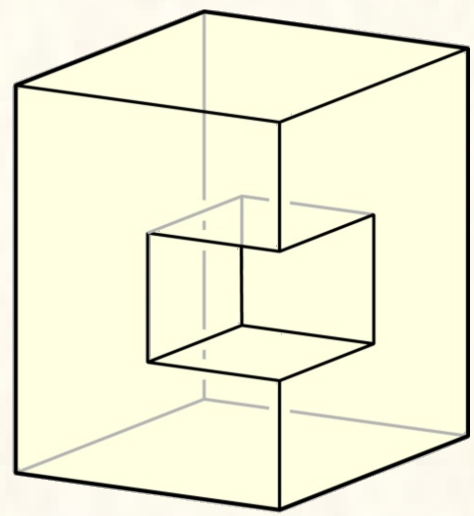
<u>are exactly</u>

Cubic bipartite planar 2-connected graphs such that the removal of any two vertices leaves at most 2 connected components

Skeletons of Simple Orthogonal Polyhedra



a graph that is NOT
a skeleton of a simple
orthogonal polyhedron



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Cubic bipartite planar 2-connected graphs
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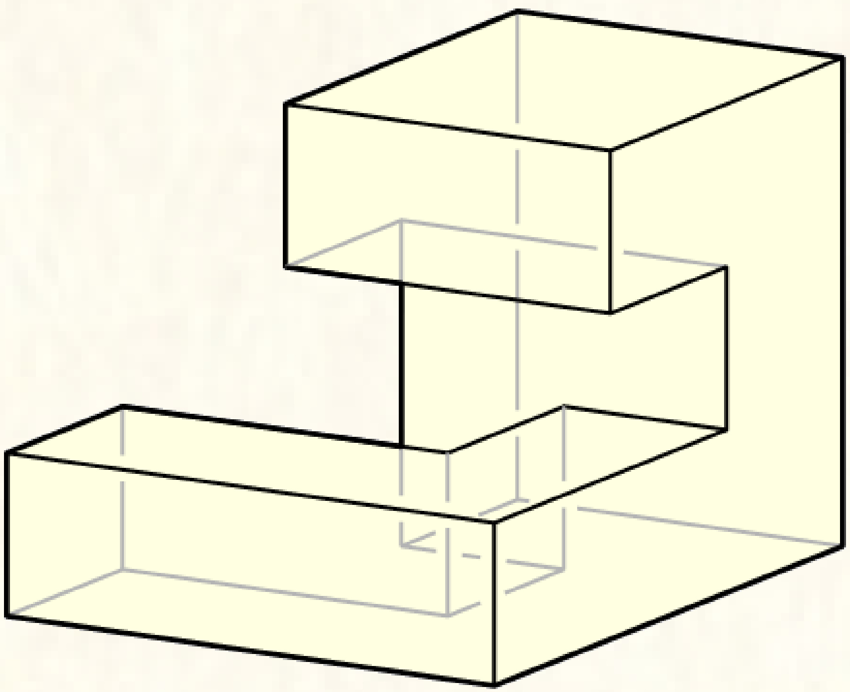
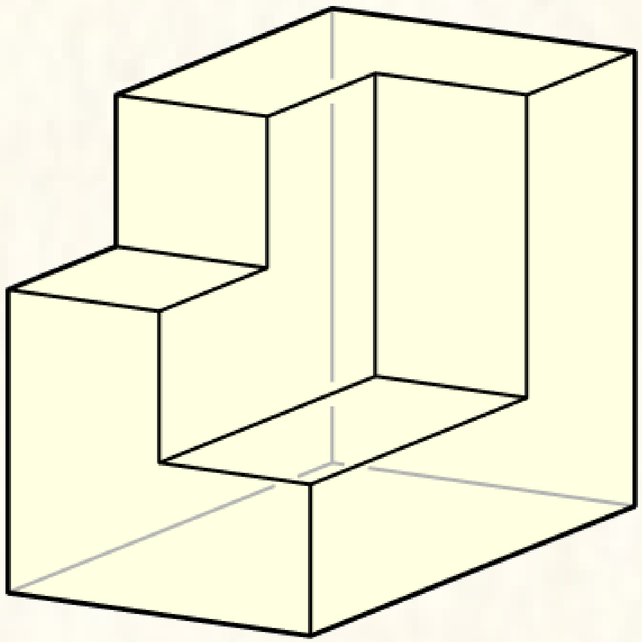
Skeletons of XYZ polyhedra

are exactly

cubic bipartite planar 3-connected graphs





Skeletons of XYZ polyhedra

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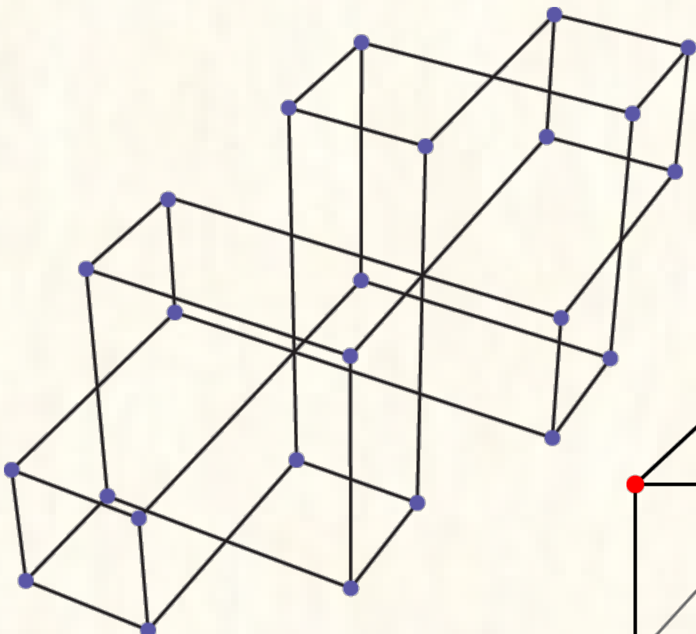
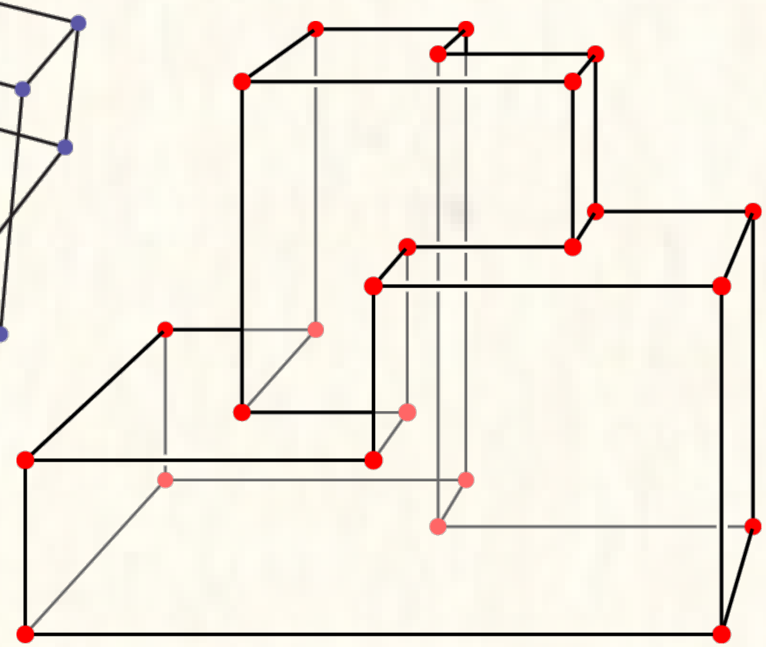
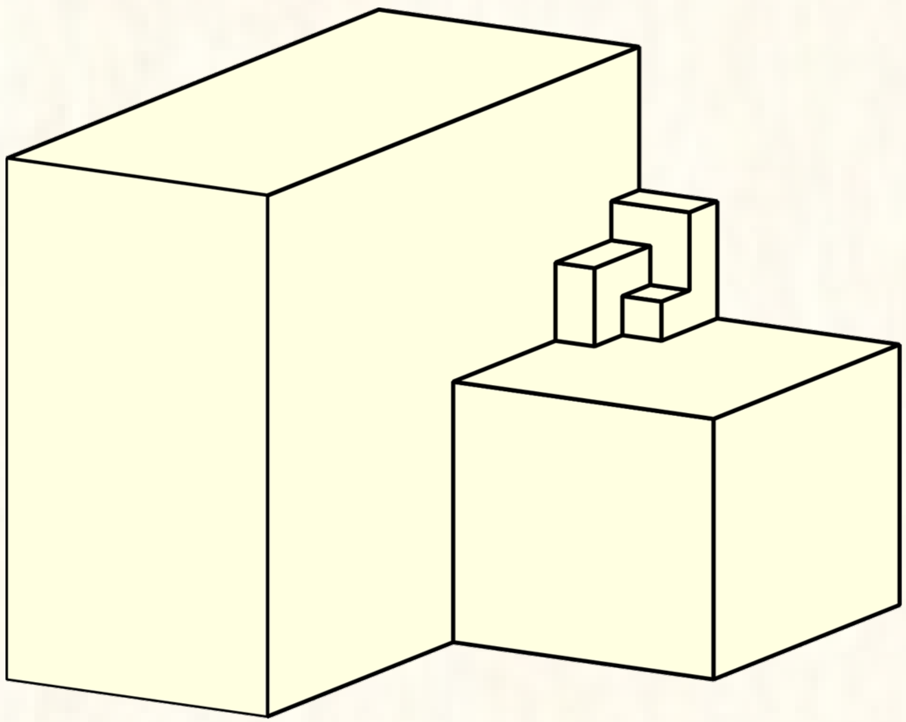


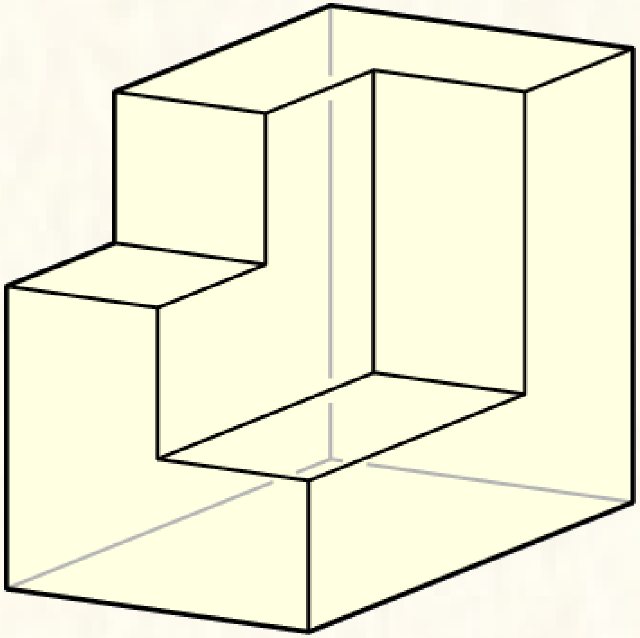


Eppstein GD'08

A planar graph G is an xyz graph if and only if G is bipartite, cubic, and 3-connected.

Skeletons of Corner Polyhedra

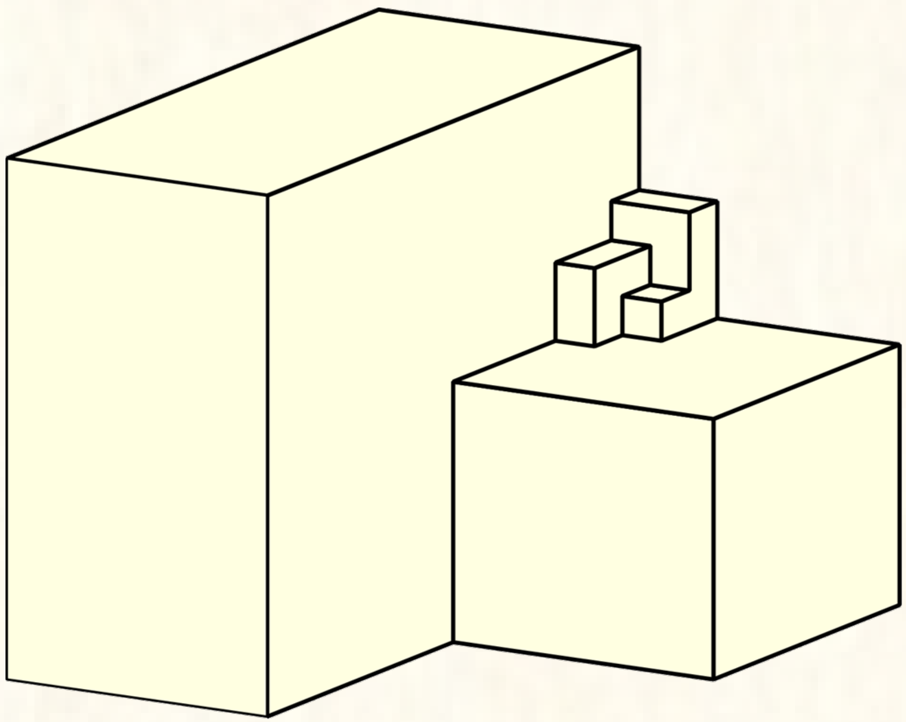


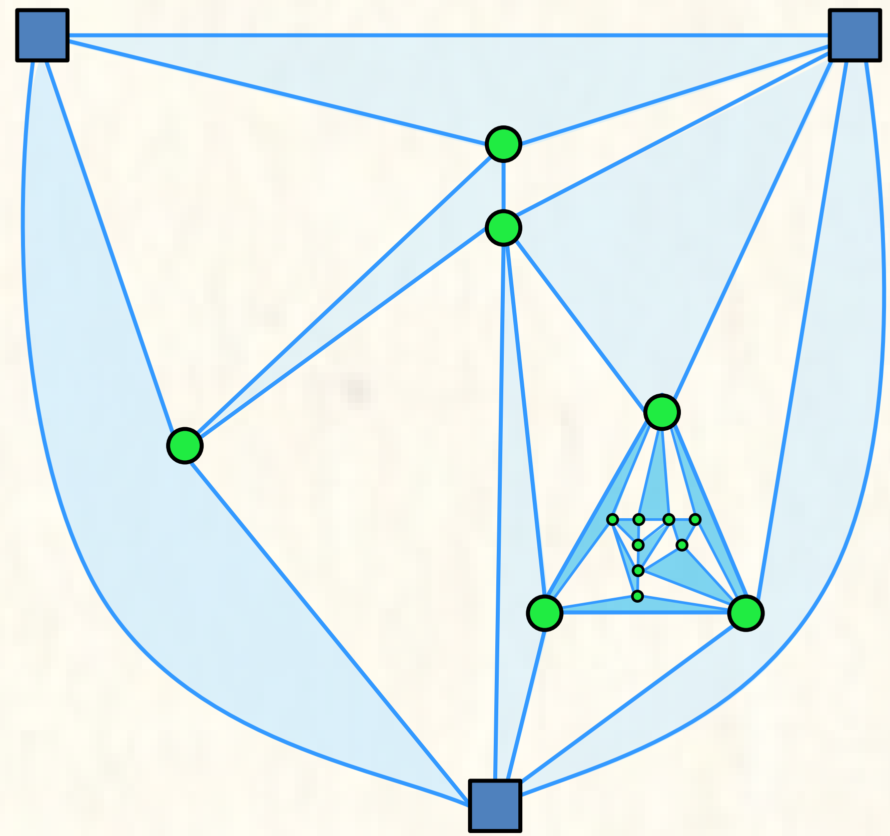


<u>are exactly</u>

cubic bipartite planar 3-connected graphs s.t. every separating triangle of the planar dual graph has the same parity.

Skeletons of Corner Polyhedra





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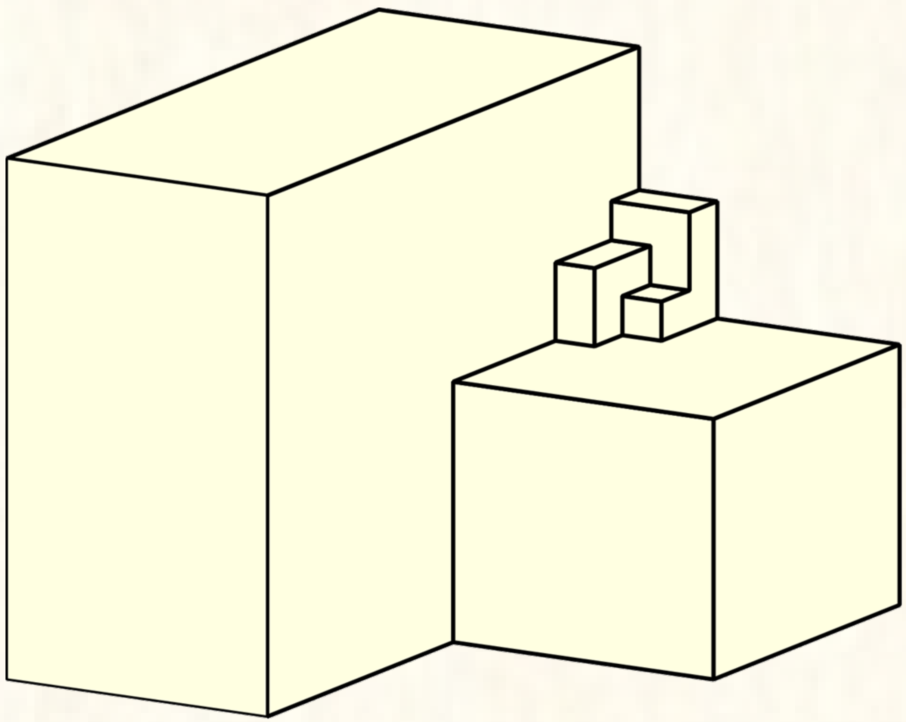

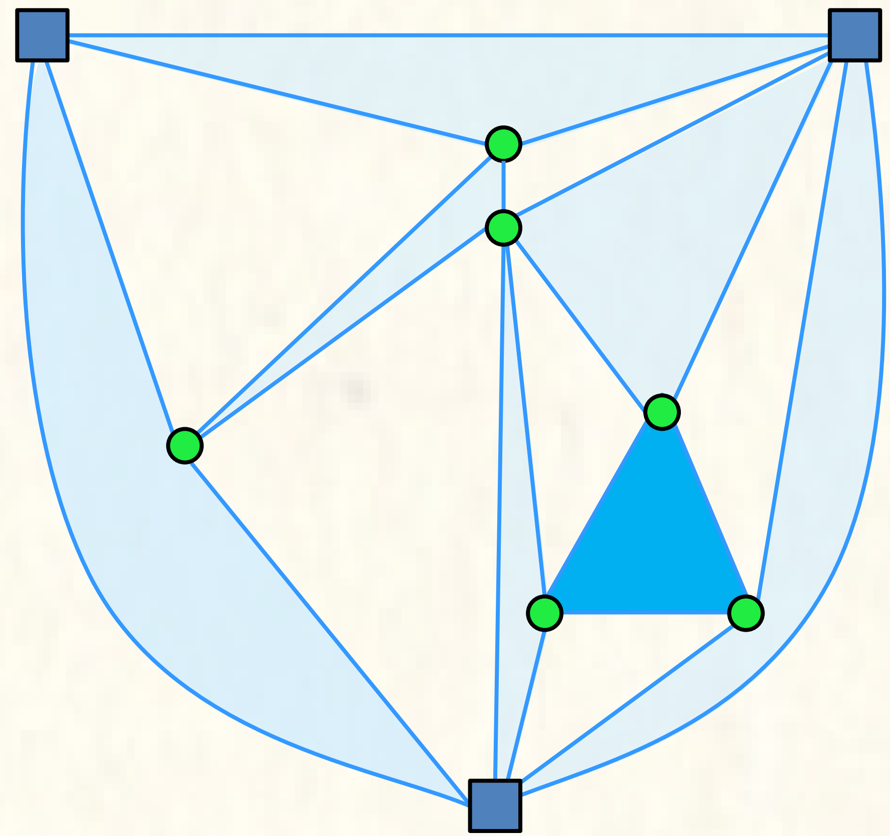

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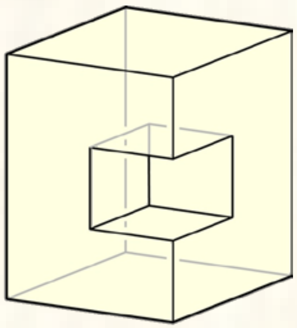
Skeletons of...



...simple orthogonal polyhedra

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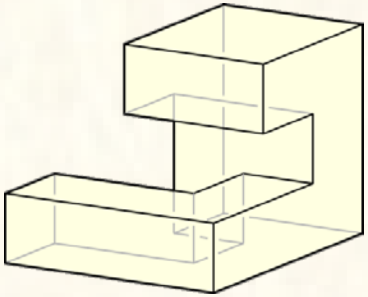
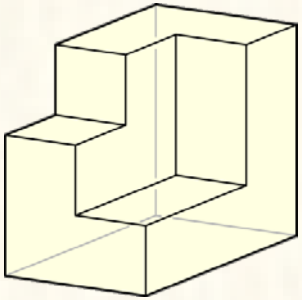


...XYZ polyhedra

cubic bipartite planar 3-connected graphs

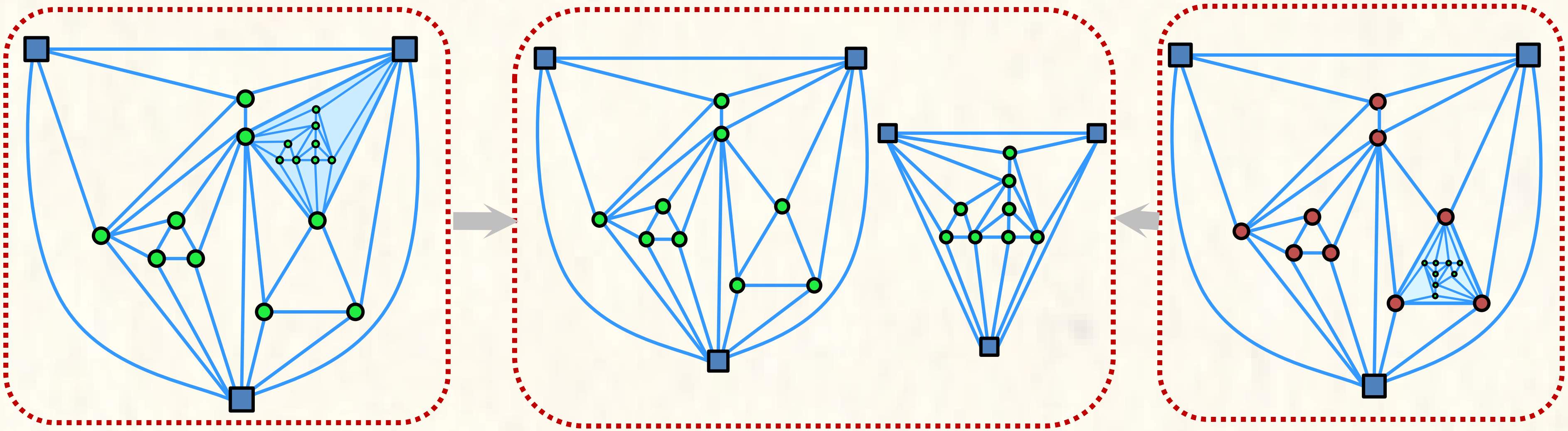


...corner polyhedra

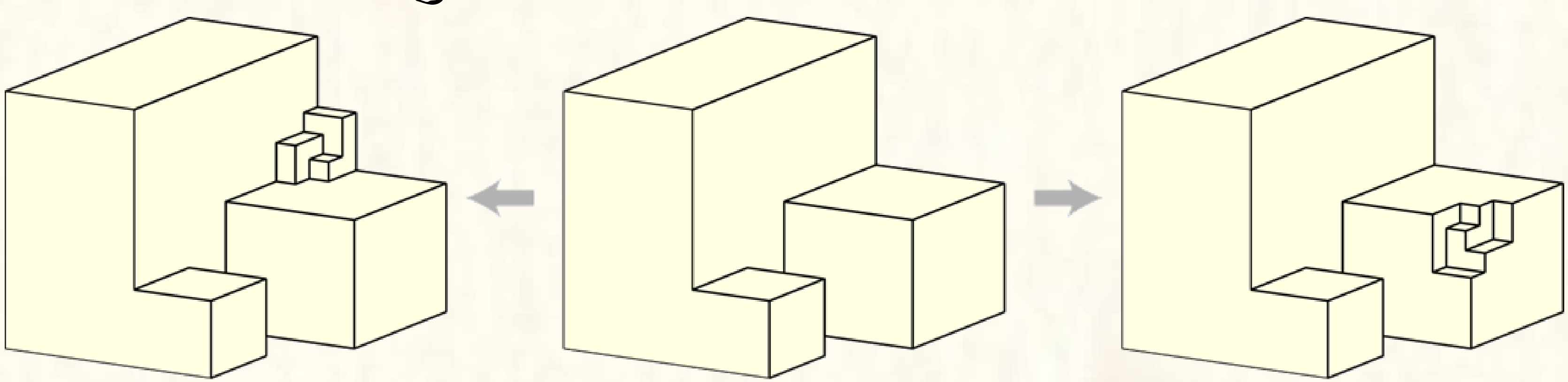
cubic bipartite planar 3-connected graphs s.t. every separating triangle of the planar dual graph has the same parity.

Rough outline for a 3-connected graph

1. Split the dual along separating triangles

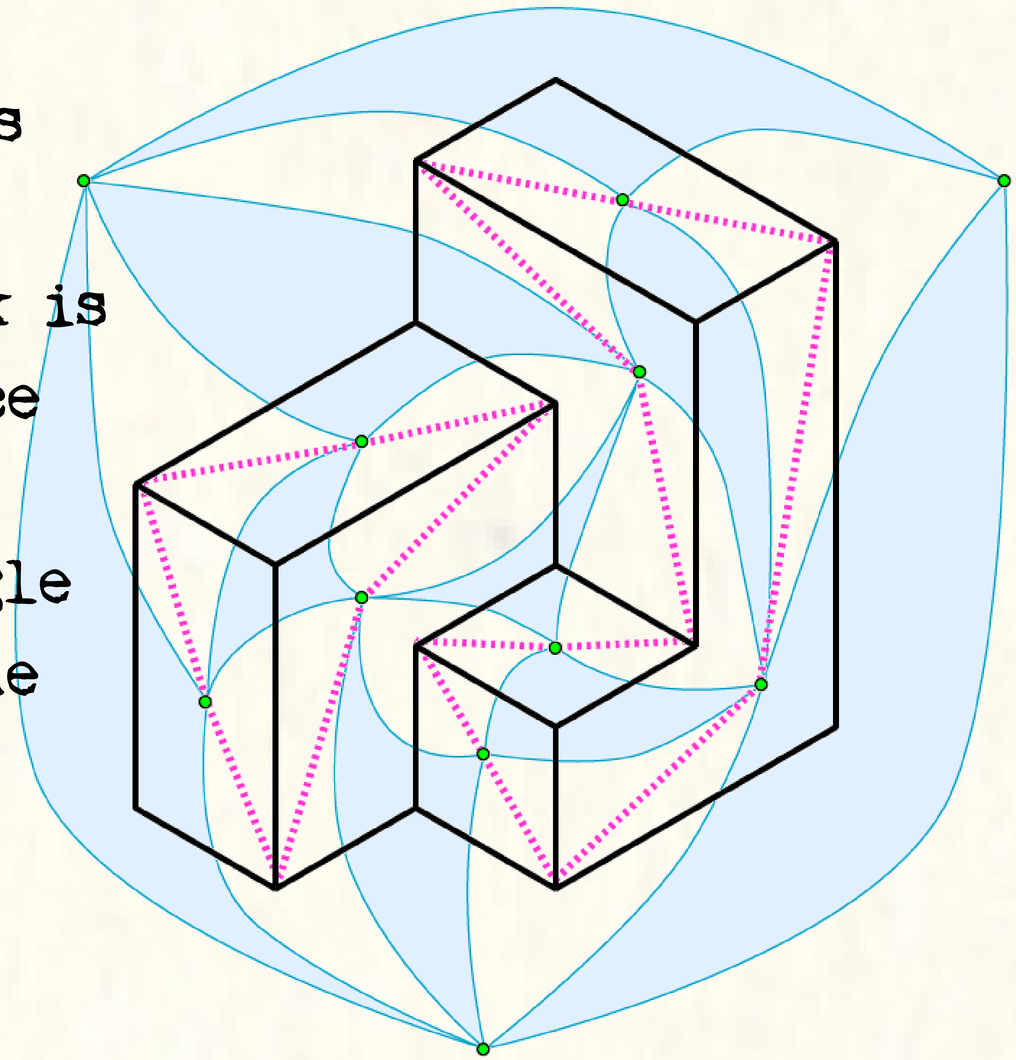


2. Construct polyhedra for 4-connected triangulations
3. Glue them together:



Rooted cycle covers

1. Collection of cycles
2. Every inner vertex is covered exactly once
3. Every white triangle contains exactly one edge of the cycle

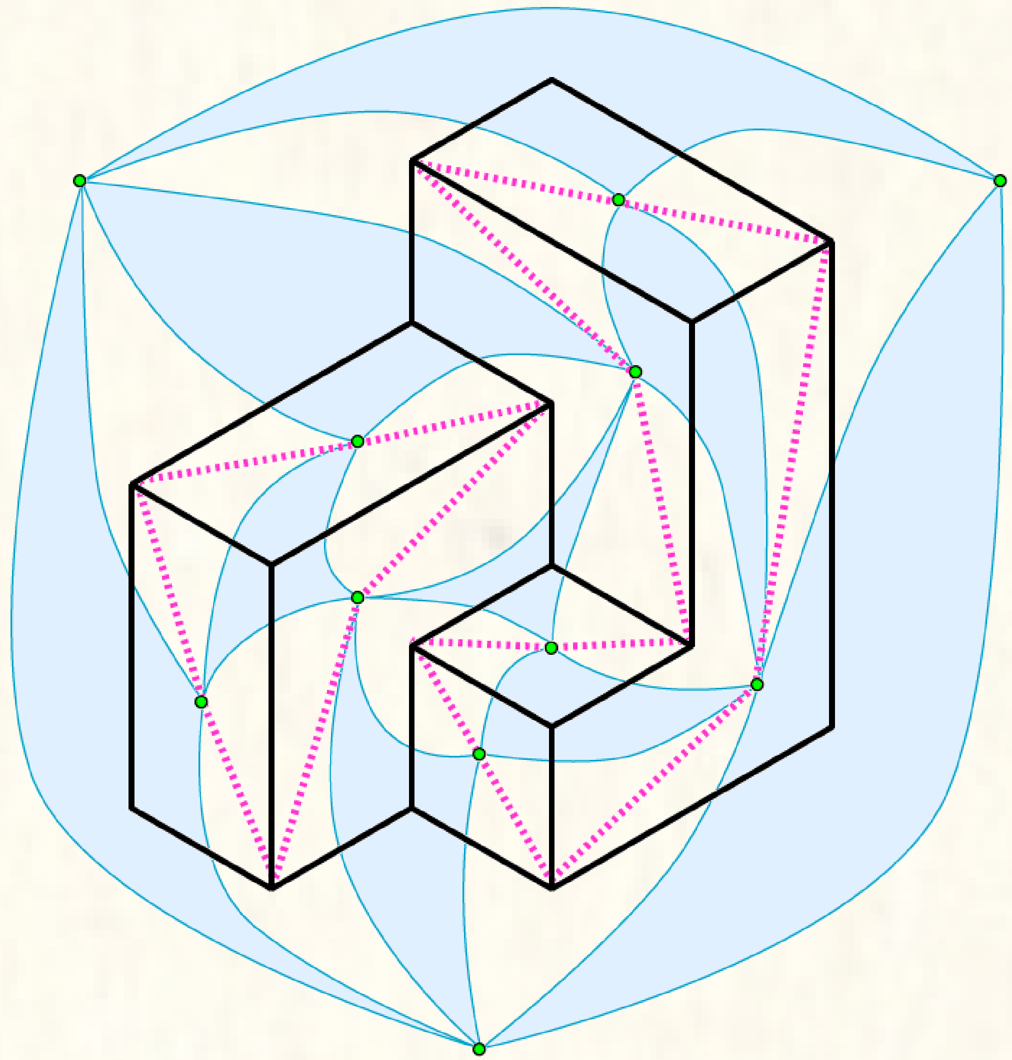


Rooted cycle covers

Rooted cycle cover

=

embedding
as a corner
polyhedron



Every 4-connected Eulerian triangulation has a rooted cycle cover

Rough outline for a 3-connected graph

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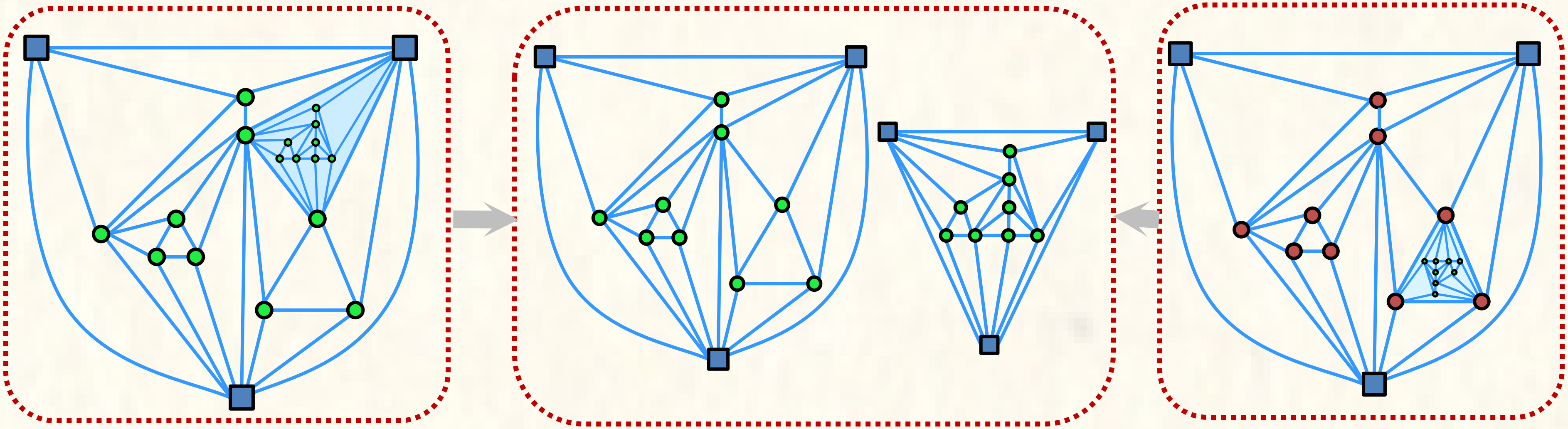

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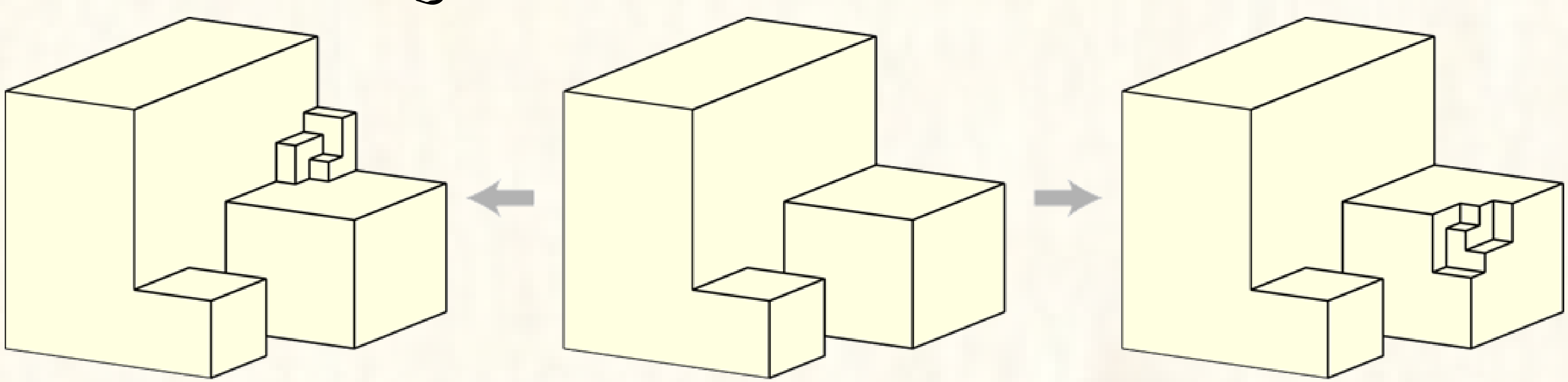

Results

- Combinatorial characterizations of skeletons of simple orthogonal polyhedra, corner polyhedra and XYZ polyhedra.

- Algorithms to test a cubic 2-connected graph for being such a skeleton in $O(n)$ randomized expected time or in $O(n (\log \log n)^2 / \log \log \log n)$ deterministically with $O(n)$ space.

- Four simple rules to reduce 4-connected Eulerian triangulation to a simpler one while preserving 4-connectivity.

Questions?