Perceptual Organization and Linear Algebra

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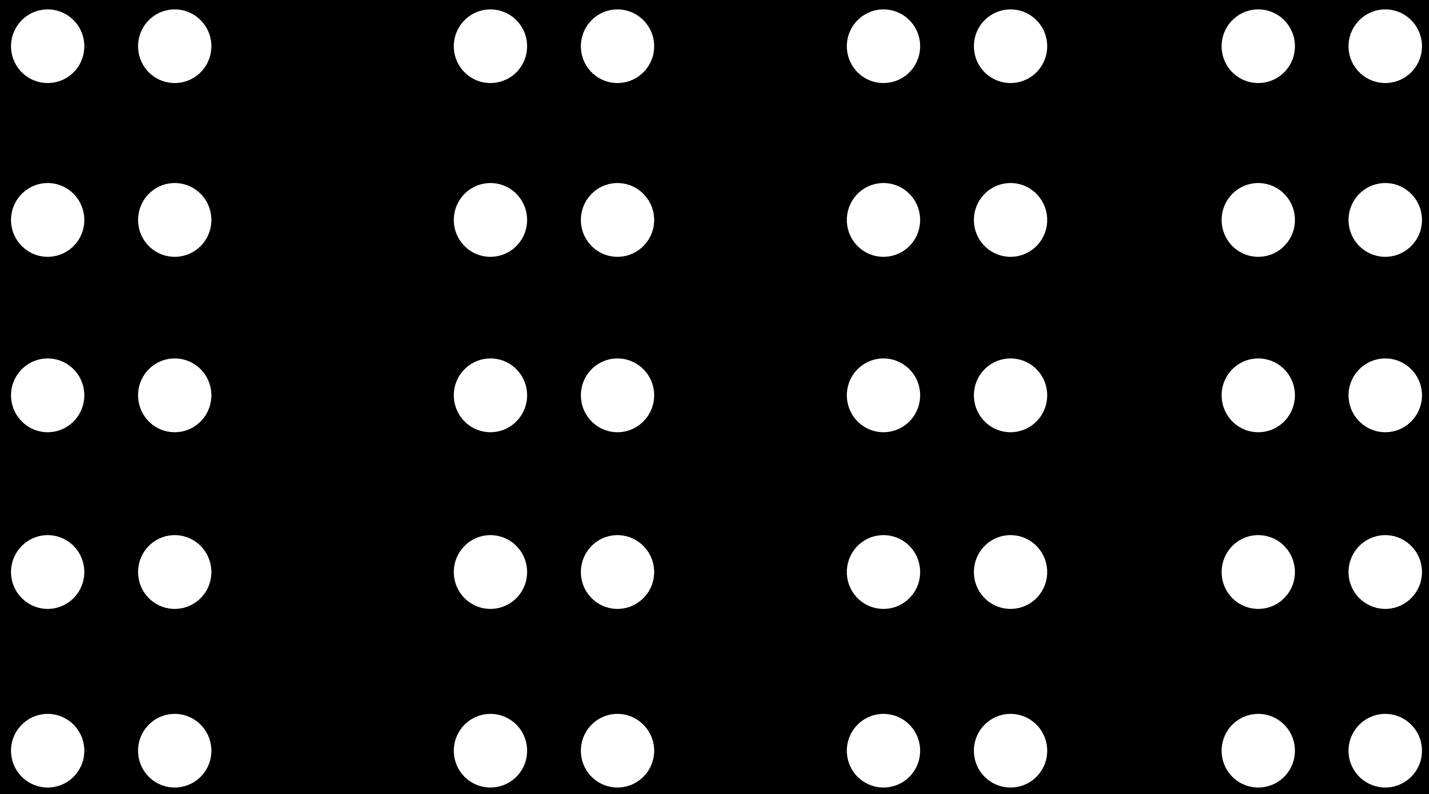
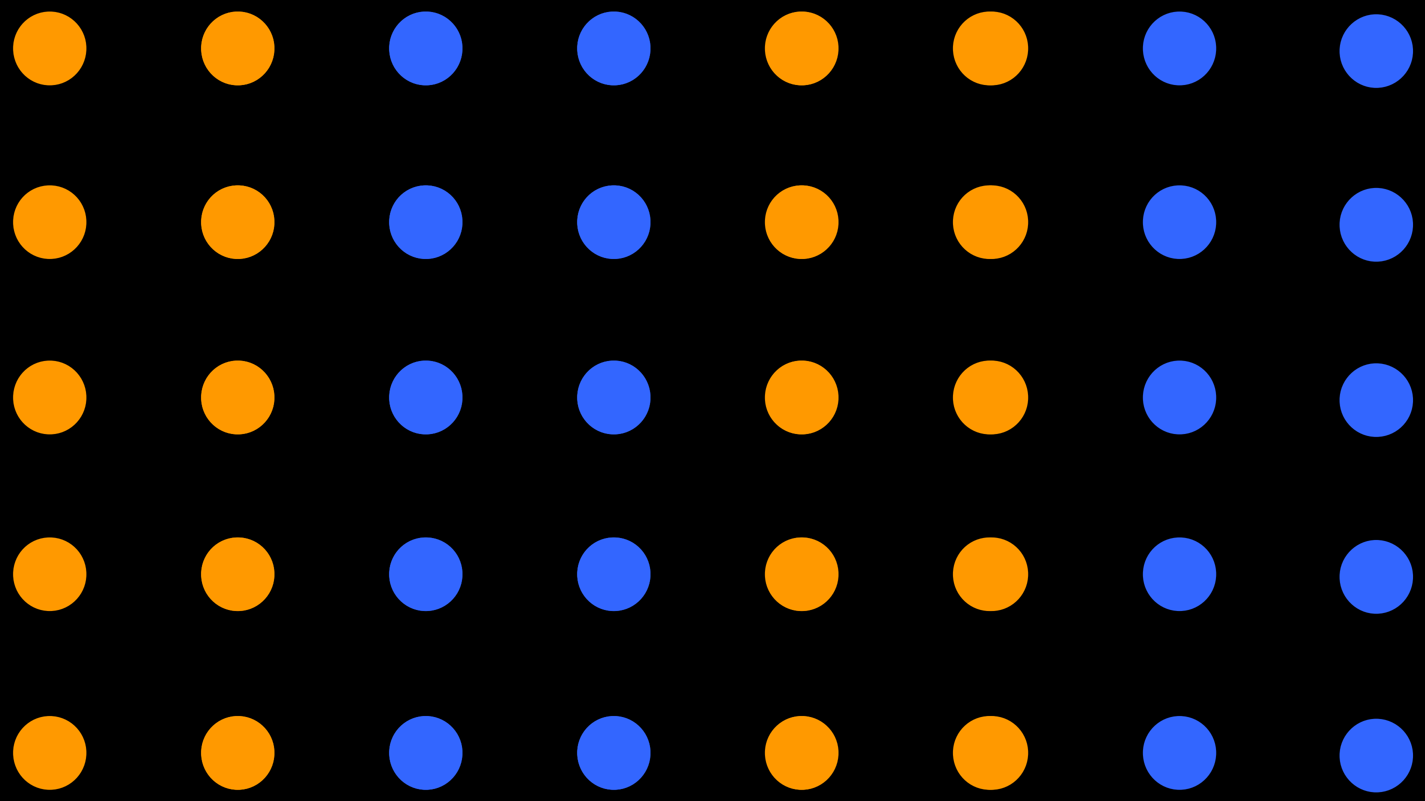
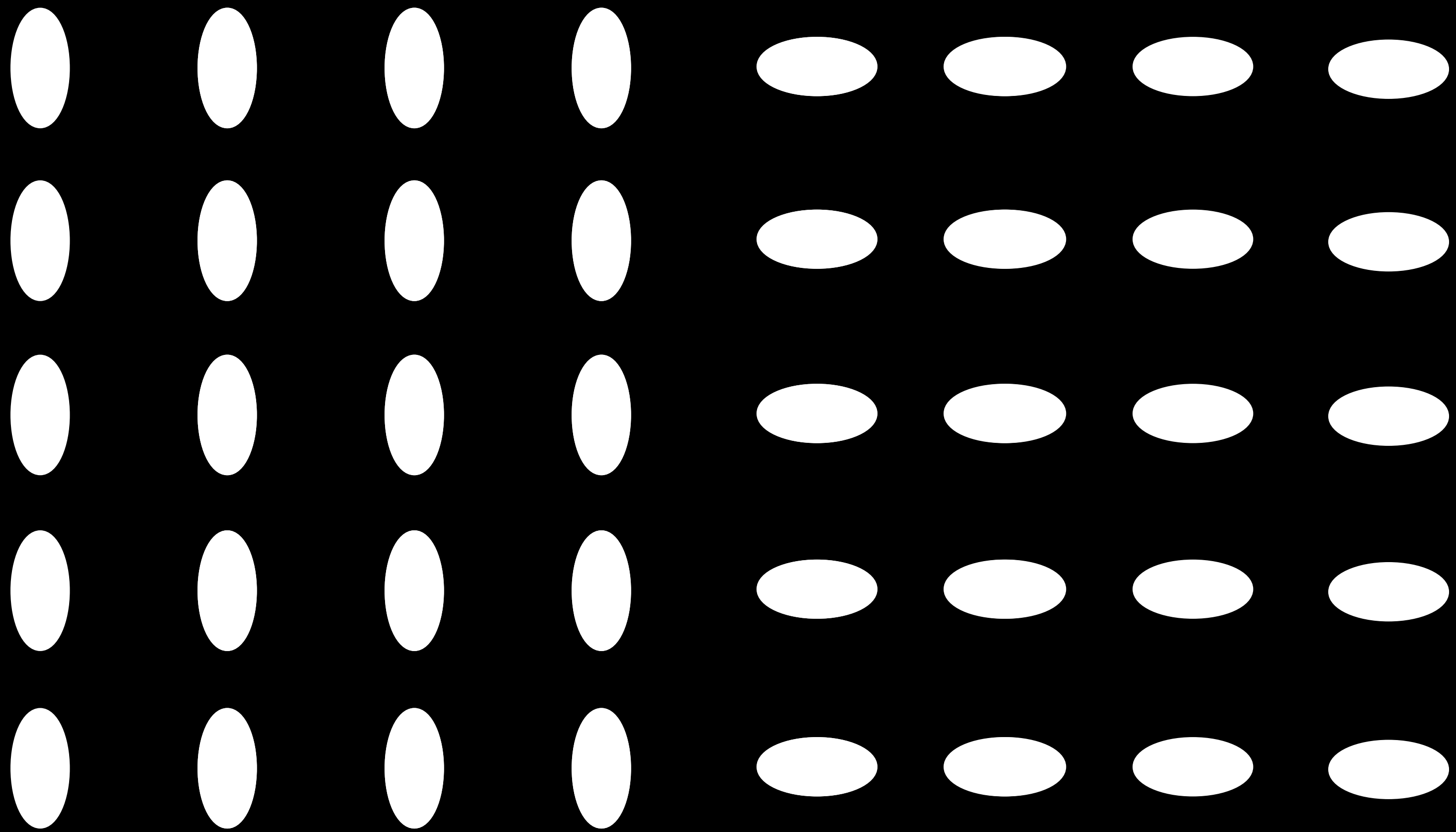
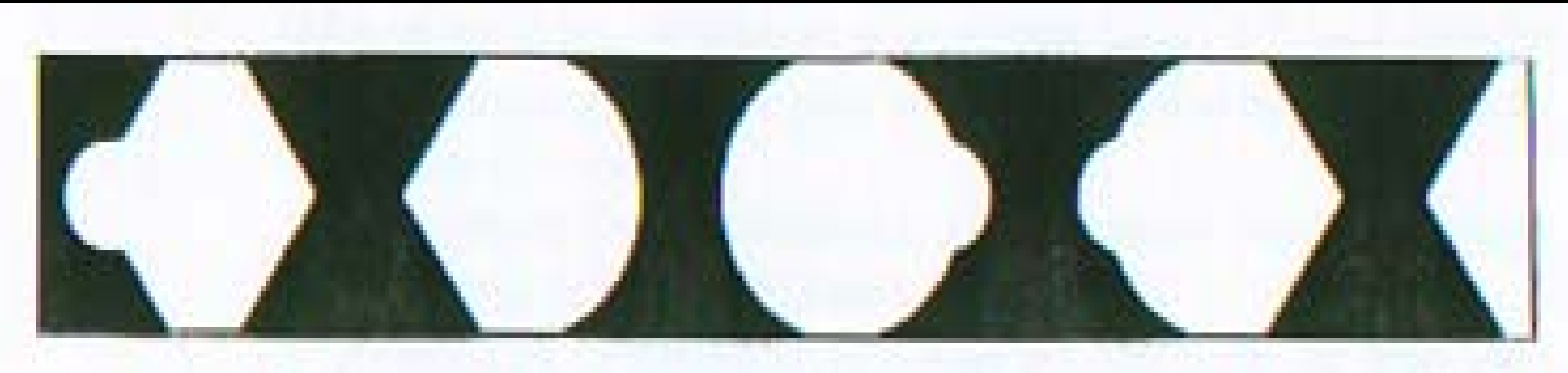


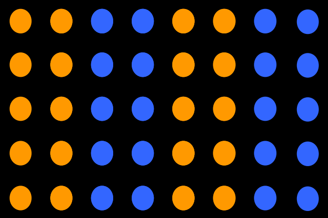






Perceptual Organization

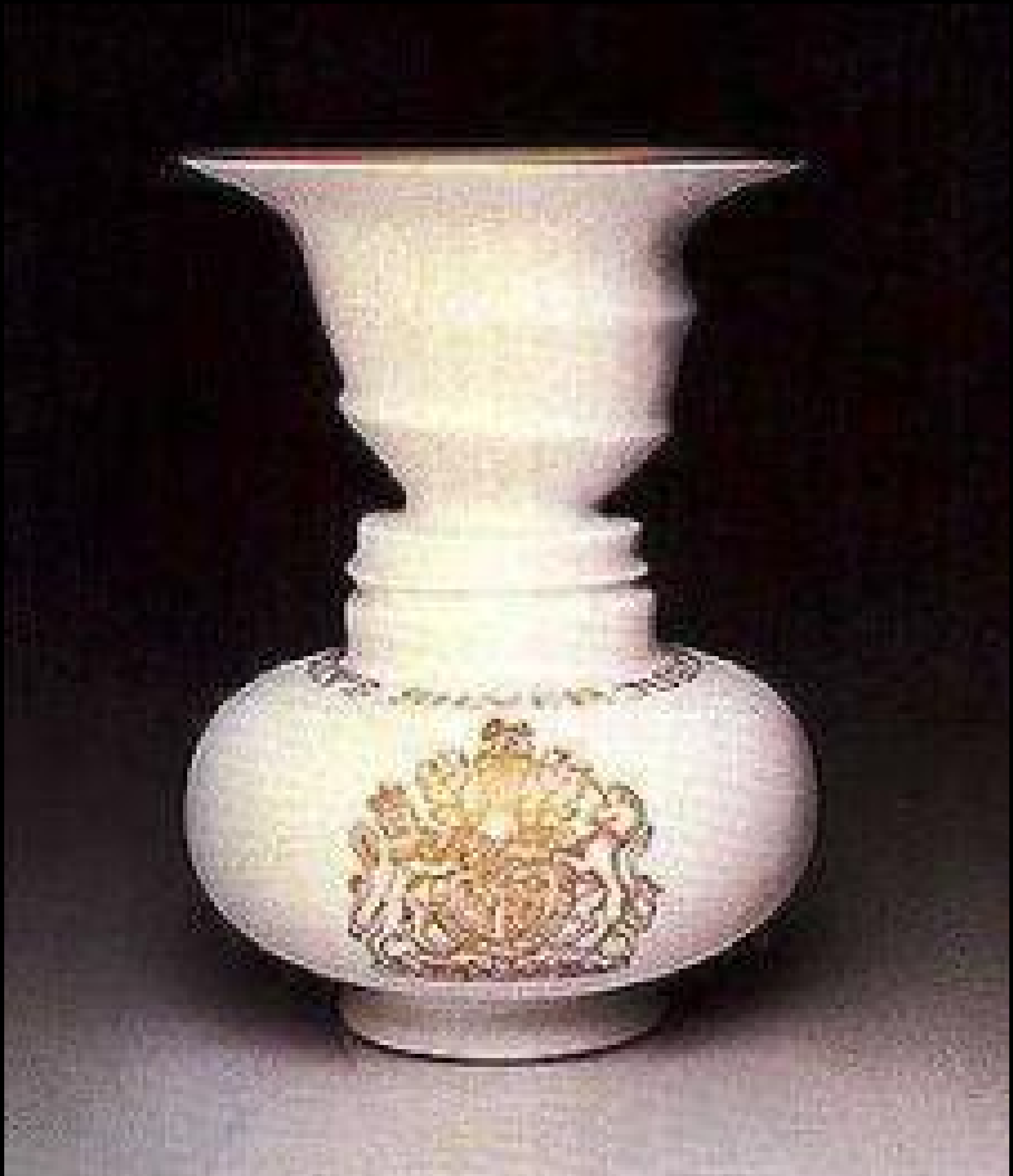
- Gestalt school of psychology (1920's)
  - *Whole is greater than the sum of its parts*
- Grouping Principles
  - Proximity, Similarity



- Figure/Ground Organization
  - Convexity, Symmetry, Familiarity

- Do these visual phenomenon matter for real images?
- Why do they exist?
- Can we build a computational model of this process?

Q: Do these visual phenomenon matter for real images?



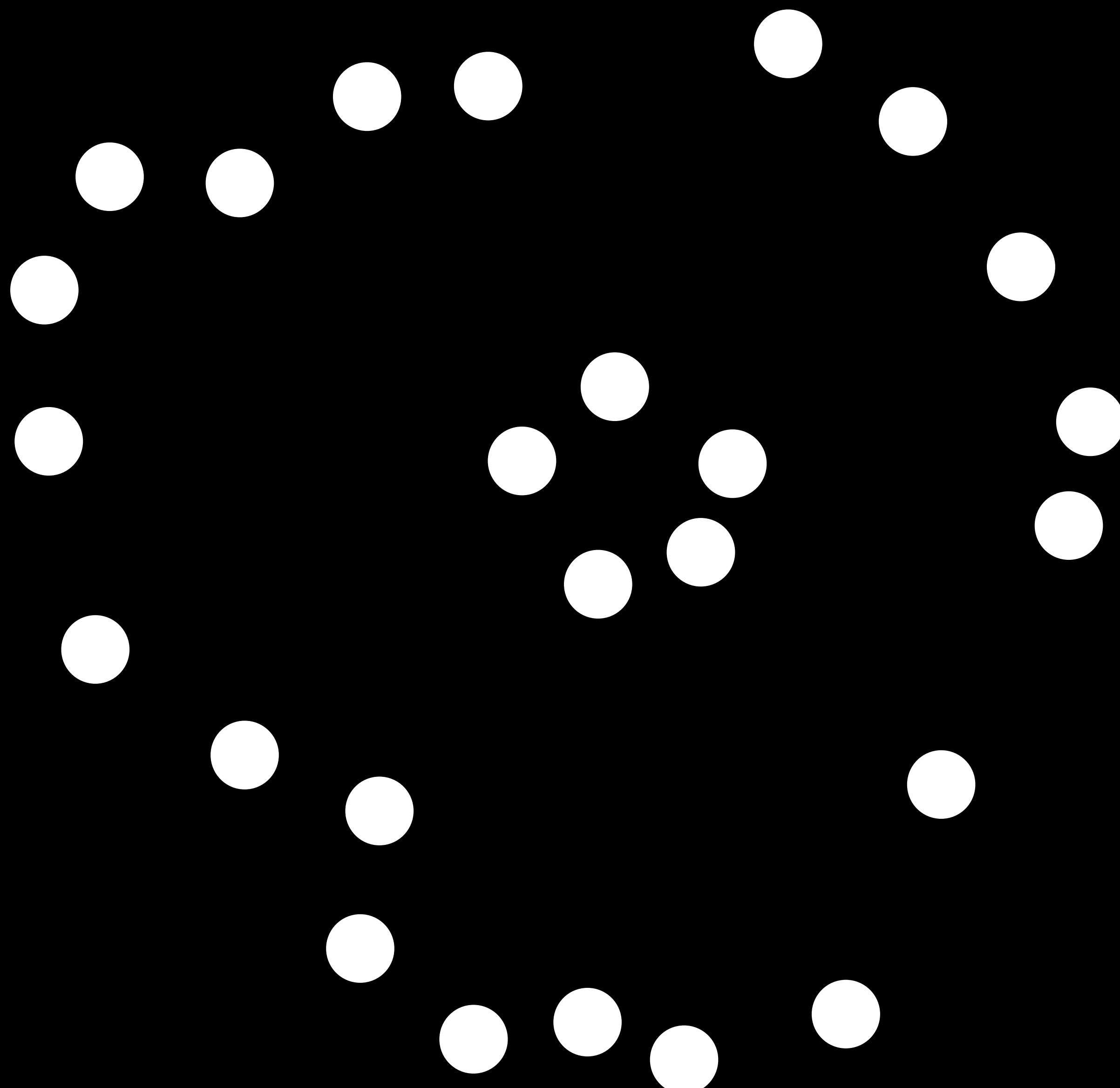
Q: Why does the brain do this?

A: Perceptual organization is a useful trick for making sense of our world.

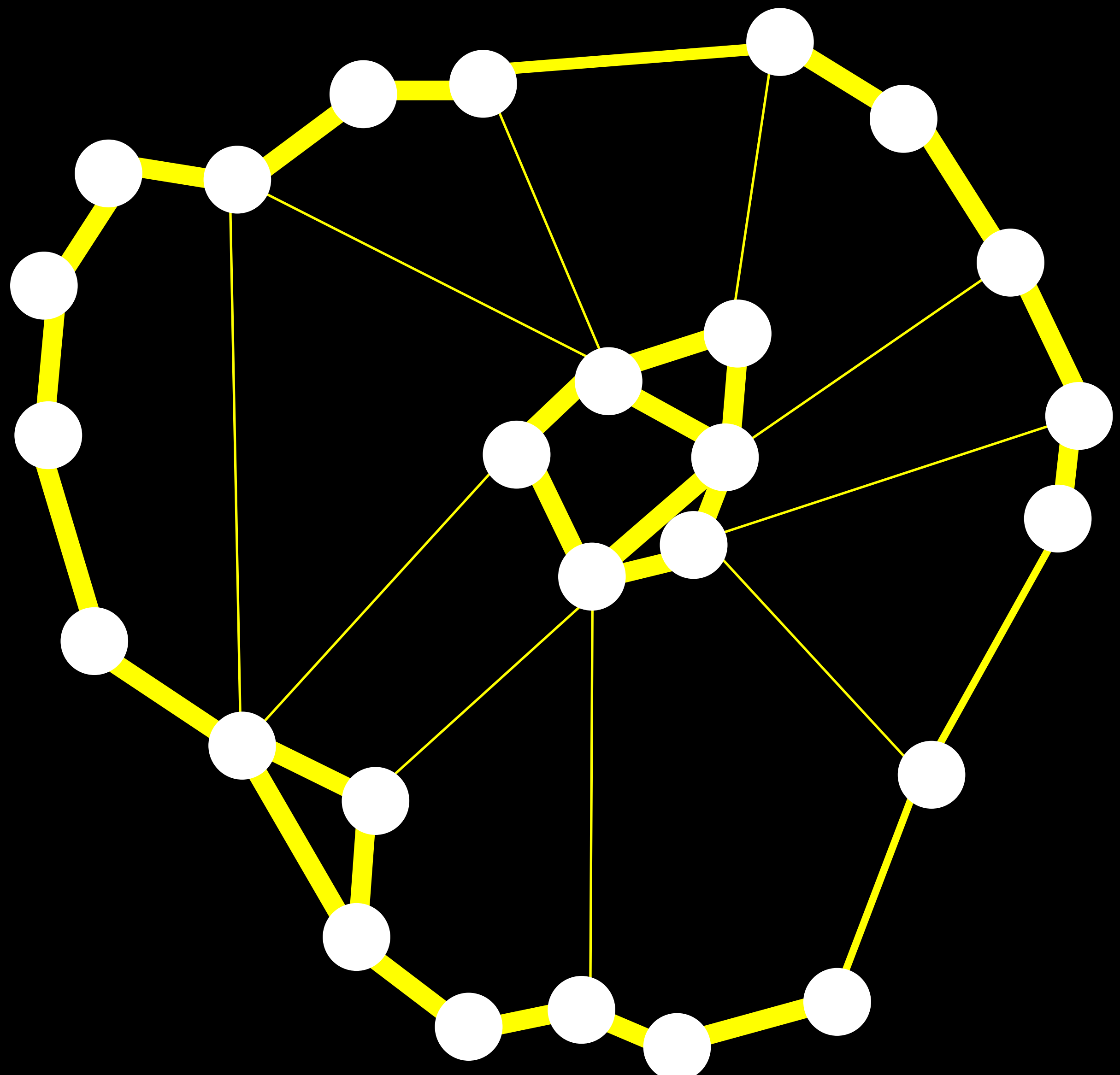
- Study the statistics of natural images.
- If similar pixels are statistically more likely to belong to the same object/surface then this all makes sense.

Q: Can we build a mathematical model of this process?

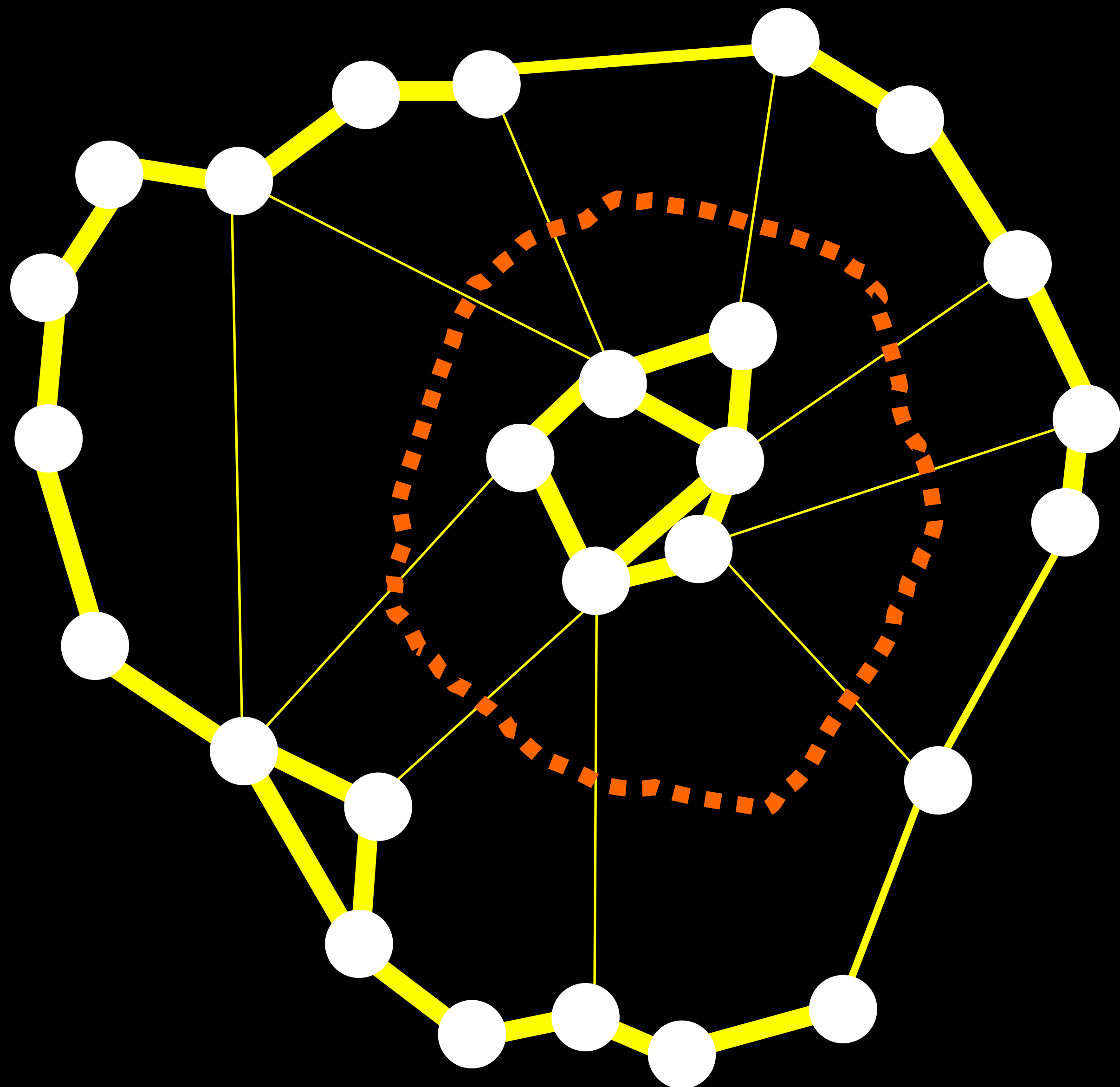
A: Let’s roll up our sleeves and give it a try!



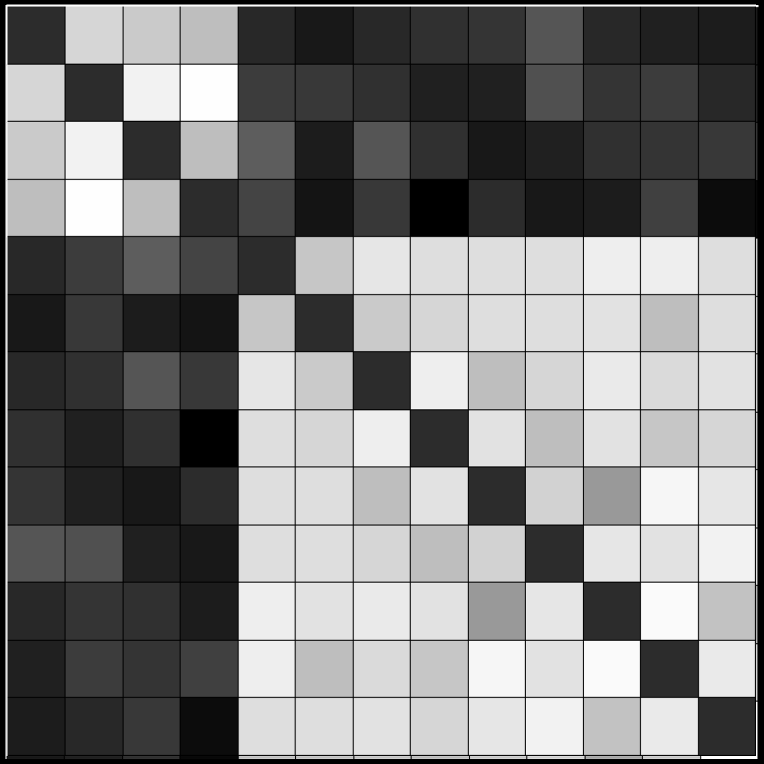
Encode Pairwise Relationships as a Weighted Graph



Cut the graph into two pieces

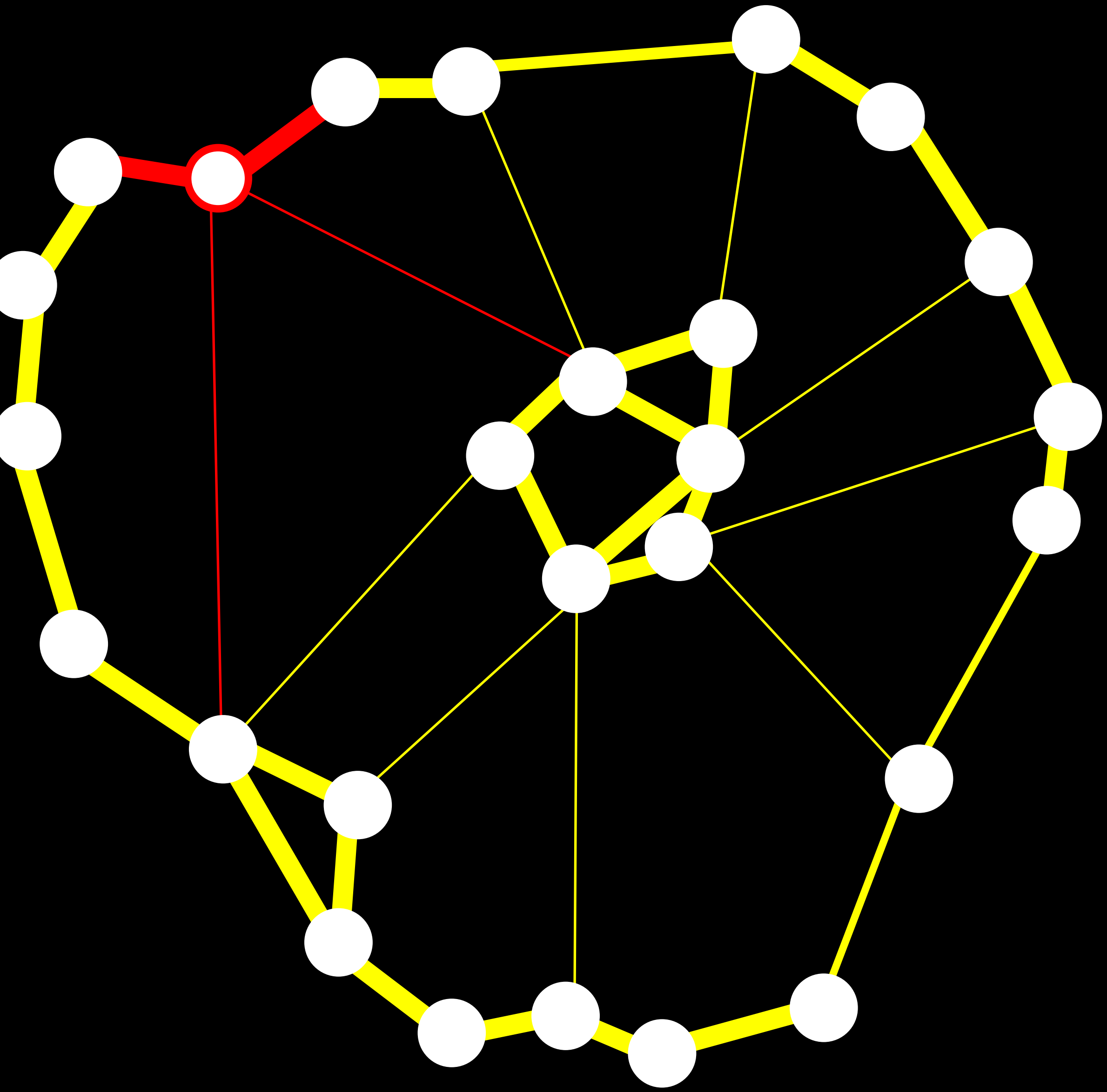


Algebraic representation of a weighted undirected graph



W(i,j) = similarity of vertex i and vertex j

$W(i,j) = W(j,i) \in [0,1]$

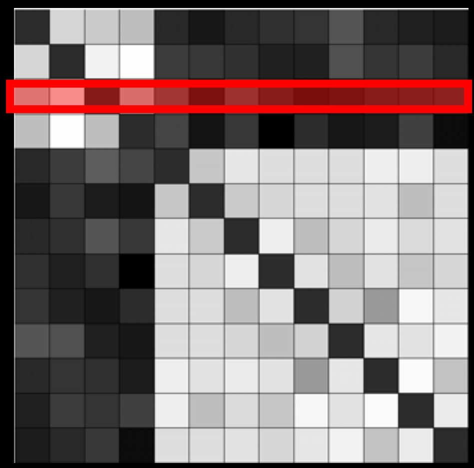


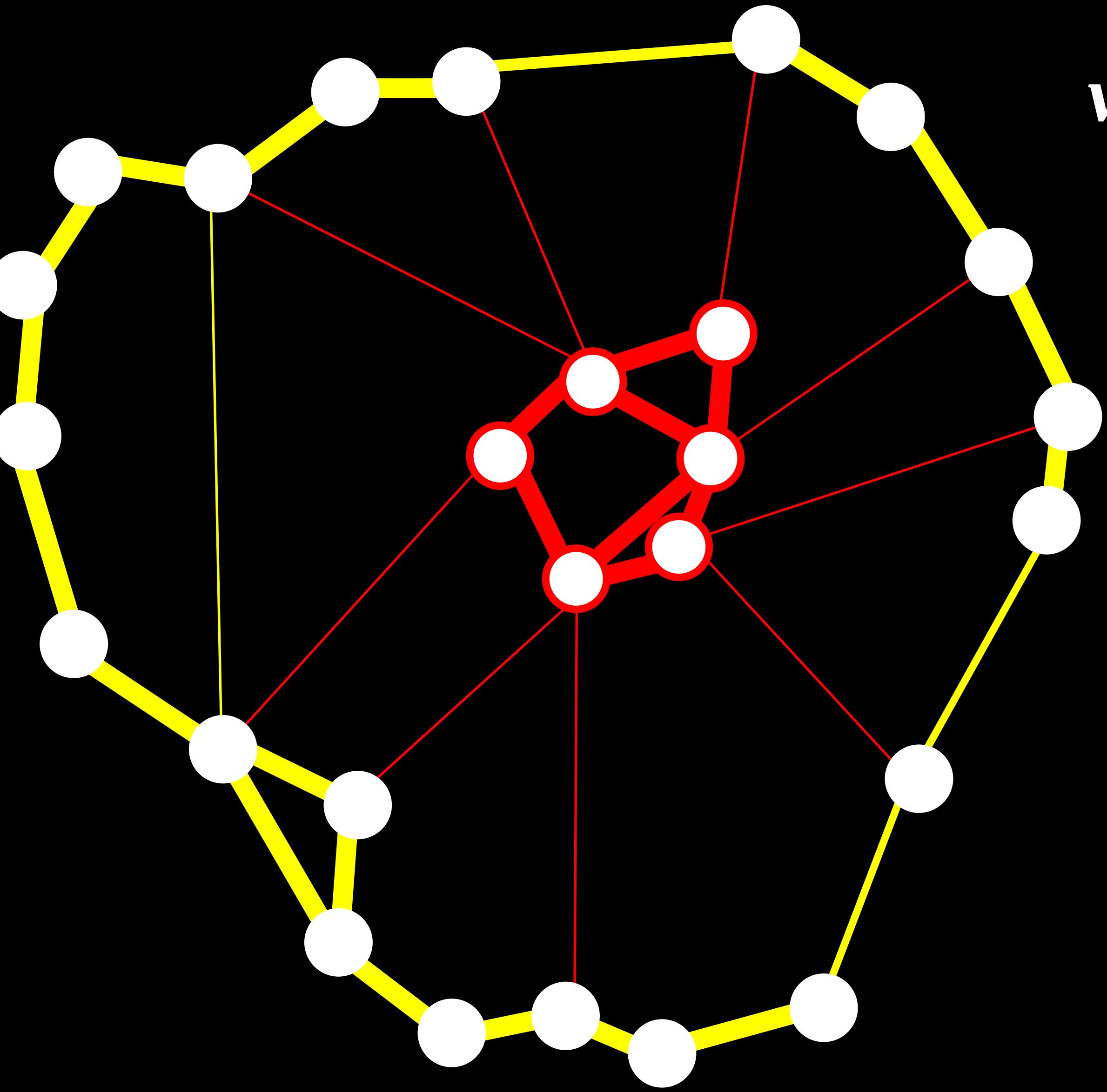
Degree of a node:

$$\mathbf{d_i = \sum_j W_{i,j}}$$

Degree matrix:

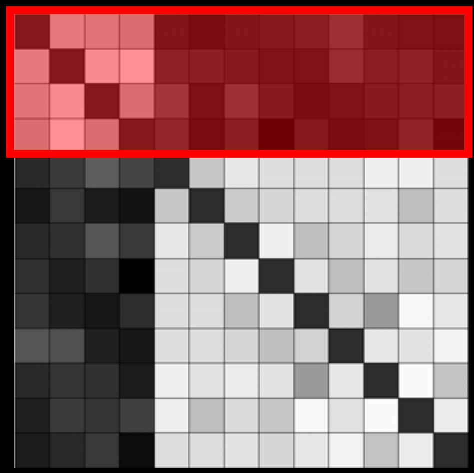
$$\mathbf{D_{ii} = \sum_j W_{i,j}}$$





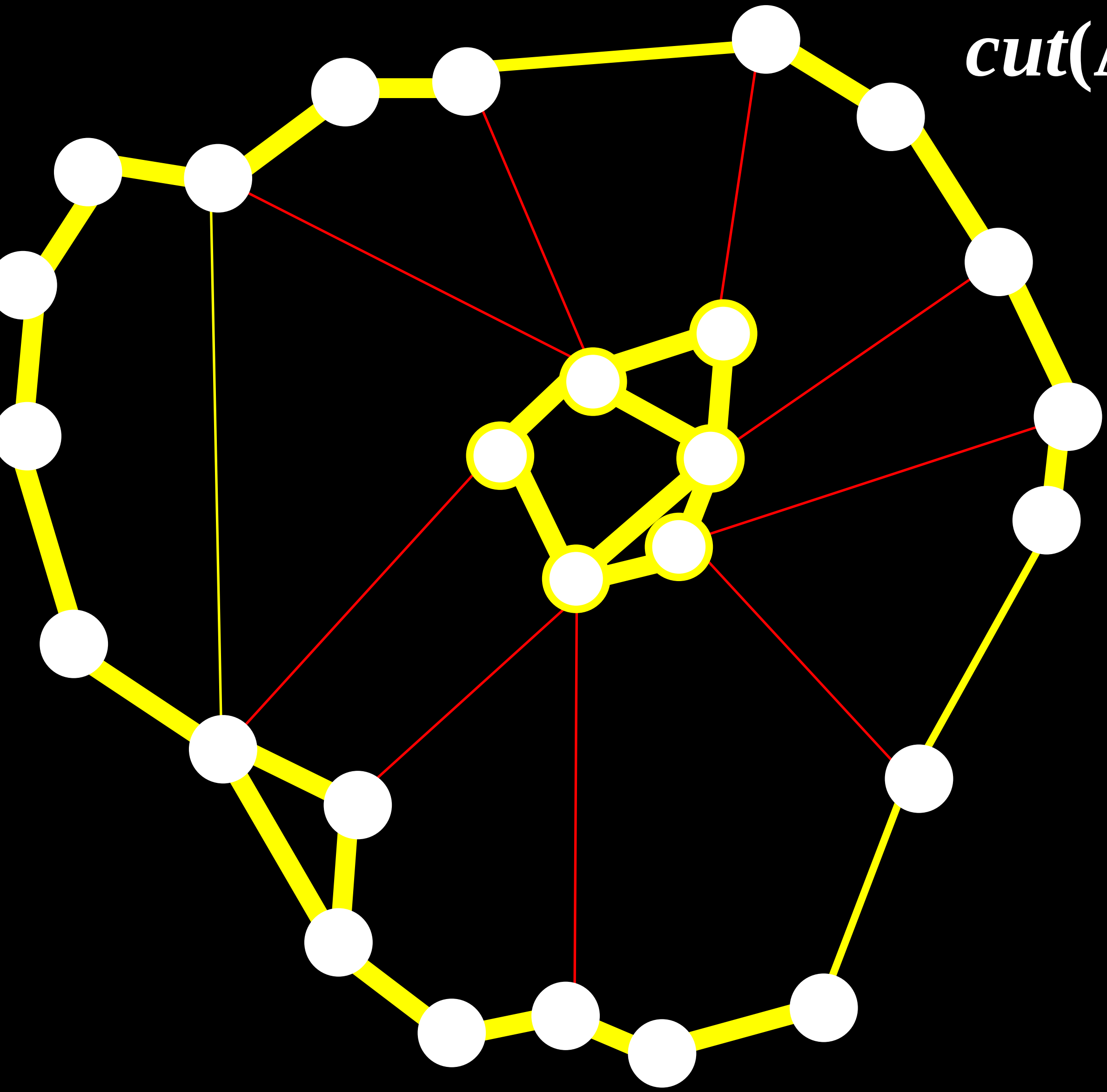
Volume of a set

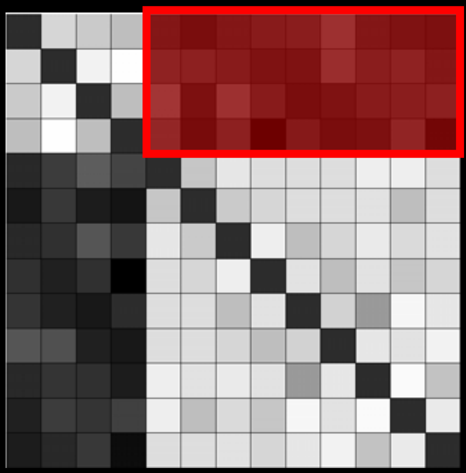
$$vol(A) = \sum_{i \in A} d_i$$



Weight of a cut:

$$cut(A,B) = \sum_{i \in A,\, j \in B} W_{i,j}$$

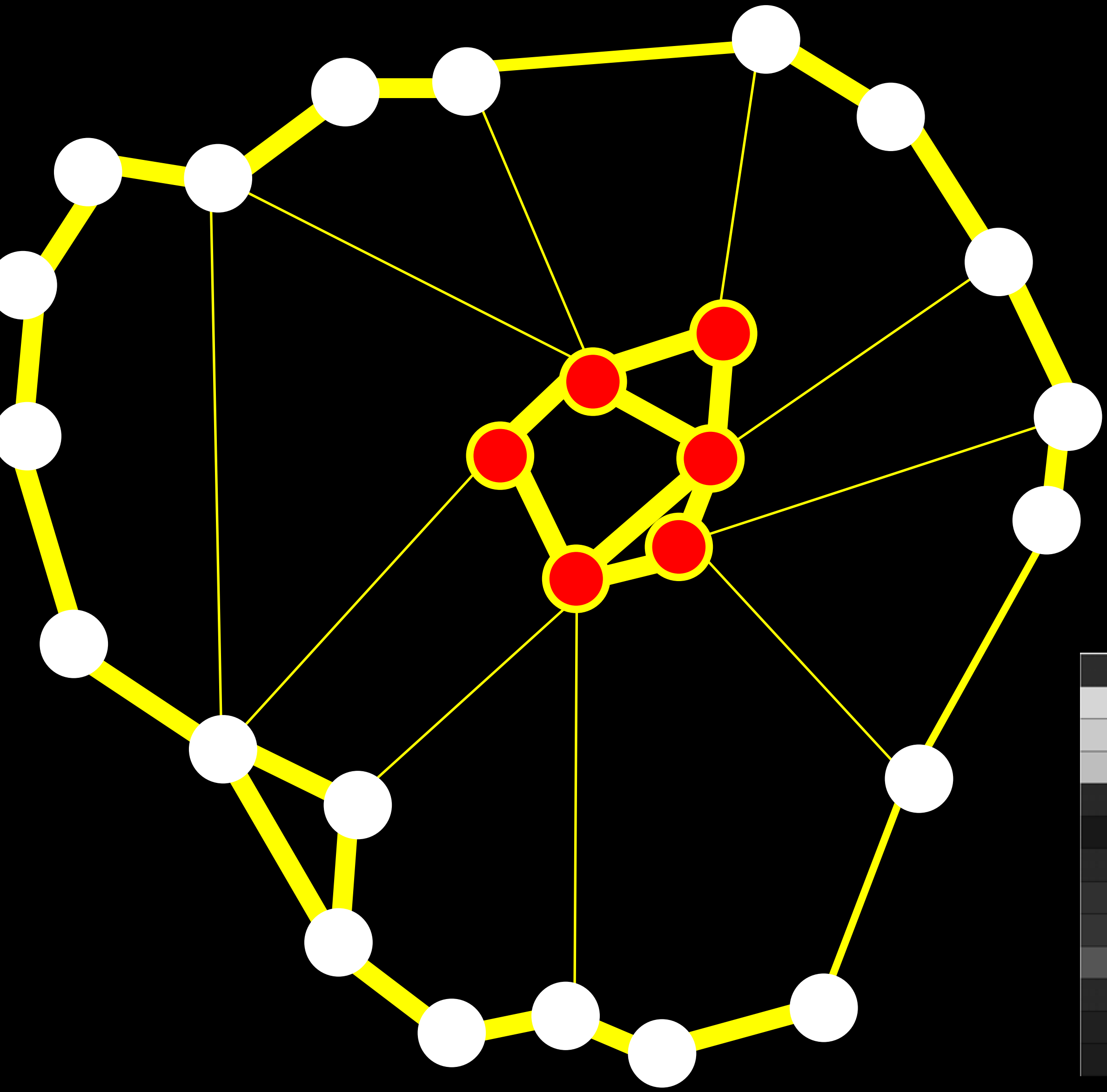


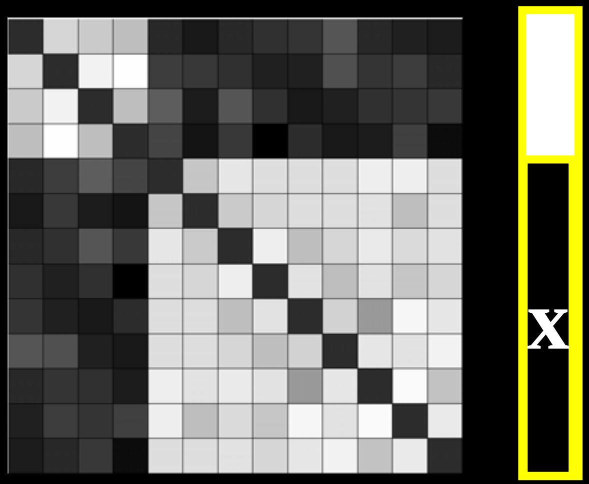


Indicator vector:

x_i = 1 if vertex i $\in$ A

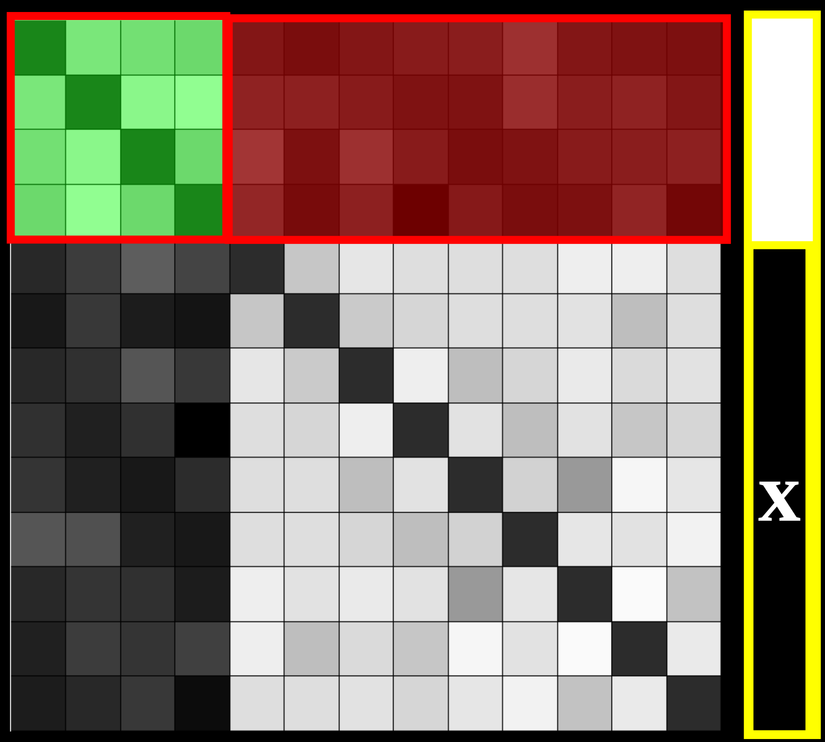
0 otherwise





$$vol(\mathrm{A}) = \sum_{i \in A} d_i \qquad cut(\mathrm{A},\mathrm{A}^c) = \sum_{i \in A,\, j \in A^c} W_{i,j}$$

$$vol(\mathrm{A}) = x^T D\, x \qquad cut(\mathrm{A},\mathrm{A}^c) = x^T (D\text{-}W)\, x$$



D-W is called the Graph Laplacian

Finding a cut with small weight

- Minimize $x^T (D-W) x$
  subject to $x \in \{0,1\}^N$

- Minimize $x^T (D-W) x$
  subject to $||x||^2=1$

This is an eigenvalue problem!

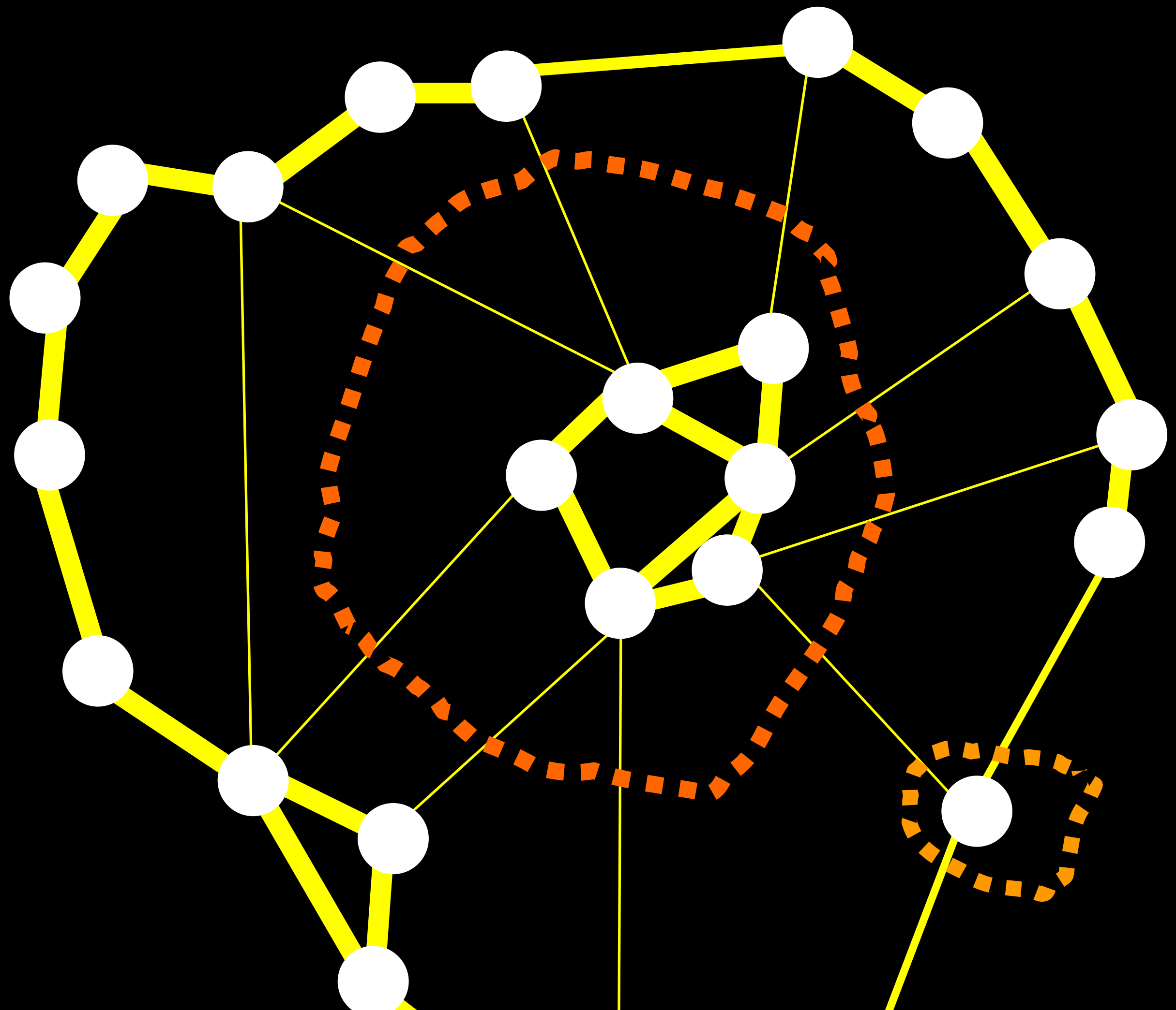
Lets solve it.

download **makeW.m** and **showGroups.m**

```
>> [W,coords] = makeSimilarity(30,70,1);
>> D = diag(sum(W,2));
>> L = D-W;
>> [U,S] = eig(L);
>> indicator = U(:,2) > 0;
>> showGroups(coords,indicator)
```

A problem with minimum cuts…

>> [W,coords] = makeSimilarity(30,70,0.5,true);



Solution: Normalized Cut

$$NCut(A, A^c) = \frac{cut(A, A^c)}{vol(A)} + \frac{cut(A, A^c)}{vol(A^c)}$$

Simultaneously minimize similarity between groups
And maximize similarity within groups

Corresponding generalized eigenvalue problem:

$$(D\text{-}W)x = \lambda Dx$$

Comparing Generalized Eigenvectors.

```
>> [W,coords] = makeSimilarity(30,70,0.5,true);
>> D = diag(sum(W,2));
>> L = D-W;
>> [U1,S] = eig(L);
>> [U2,S] = eig(L,D);

>> showGroups(coords, U1(:,2) > 0 )

>> showGroups(coords, U2(:,2) > 0 )
```

Working with real images

- Each pixel is a vertex of the graph
- Similarities between pixels correspond to weights
- For example, difference in grayscale value:

$$W(i,j) = \exp(-(I(i) - I(j))^2 / \sigma^2)$$

- Partitioning the graph gives a segmentation of the image

Working with real images

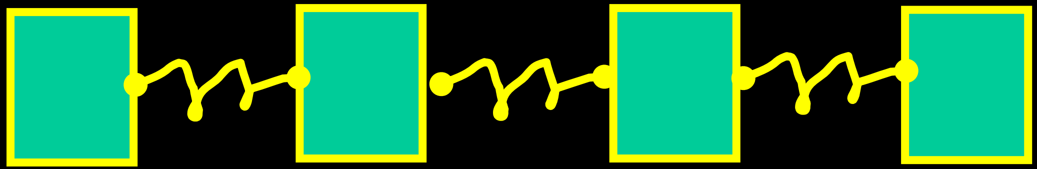
demo2.m:

```
W = makeW2(0.01,0.3);

D = diag(sum(W,1));
L = D - W;
[U,S] = eig(L,D);
im2 = reshape(U(:,2),size(im,1),size(im,2));

figure(2);
subplot(2,1,1); imagesc(im2); axis image;
subplot(2,1,2); imagesc(im2>0); axis image;
```

Generalized eigenvalue problem has interpretation as a mass-spring system $(D-W)x = \lambda Dx$



$$M\frac{d^2x}{dt^2} = -Kx$$

K is the stiffness matrix, M is the mass matrix

$$x(t) = v cos(\omega t + \phi)$$

$$\omega^2 Mv = Kv$$

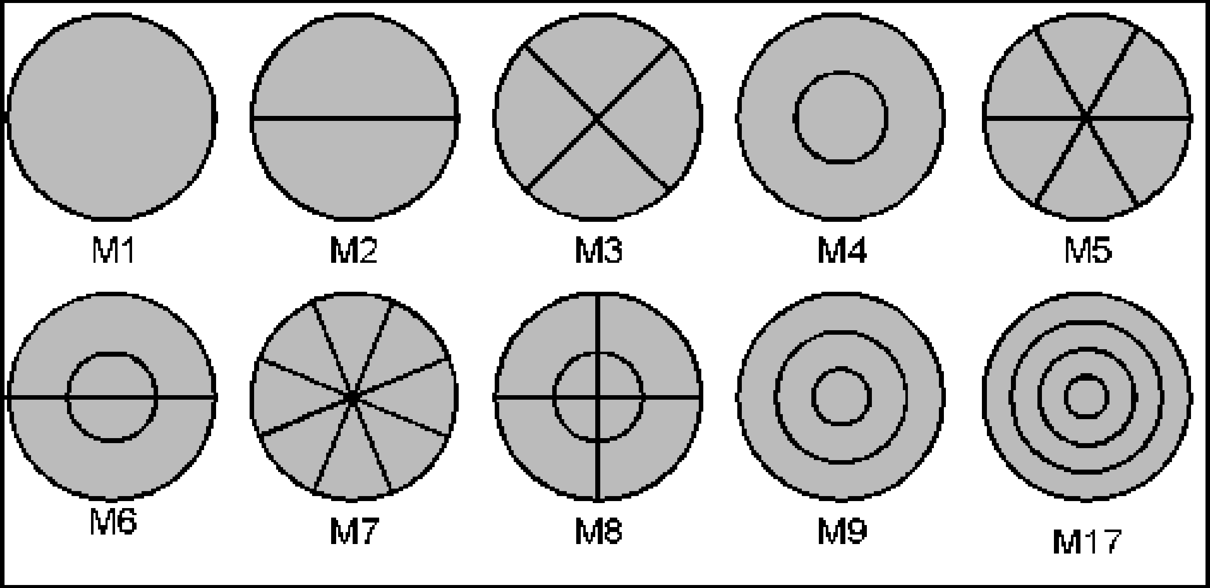
Videos

More uses of the Laplacian

- For un-weighted graphs, the combinatorial Laplacian, D-A, captures connectivity of the graph
  - det(D-A) counts # of spanning trees
  - rank(D-A) = N - #components = (0^{th} Betti number)

- Spectrum of Laplacian operator on a manifold gives information about the shape of the manifold

Can you hear the shape of a drum?



Can you hear the shape of a drum?