More NP-complete Problems
Theorem: (proven in previous class)

If: Language \( A \) is NP-complete
Language \( B \) is in NP
\( A \) is polynomial time reducible to \( B \)

Then: \( B \) is NP-complete
Using the previous theorem, we will prove that 2 problems are NP-complete:

- Vertex-Cover
- Hamiltonian-Path
Vertex Cover

Vertex cover of a graph is a subset of nodes $S$ such that every edge in the graph touches one node in $S$.

Example:

$S = \text{red nodes}$
Size of vertex-cover
is the number of nodes in the cover

Example: $|S|=4$
Corresponding language:

\[
\text{VERTEX-COVER} = \{ \langle G, k \rangle : \text{graph } G \text{ contains a vertex cover of size } k \} 
\]

Example:

\[
\langle G', 4 \rangle \in \text{VERTEX-COVER}
\]
Theorem: VERTEX-COVER is NP-complete

Proof:

1. VERTEX-COVER is in NP
   Can be easily proven

2. We will reduce in polynomial time
   3CNF-SAT to VERTEX-COVER
   (NP-complete)
Let $\varphi$ be a 3CNF formula with $m$ variables and $l$ clauses.

Example:

$\varphi = (x_1 \lor x_2 \lor x_3) \land (x_1 \lor \overline{x_2} \lor x_4) \land (x_1 \lor x_3 \lor x_4)$

Clause 1  Clause 2  Clause 3

$m = 4$

$l = 3$
Formula $\varphi$ can be converted to a graph $G$ such that:

$\varphi$ is satisfied if and only if $G$ contains a vertex cover of size $k = m + 2l$.
\[ \varphi = (x_1 \lor x_2 \lor x_3) \land (\overline{x_1} \lor \overline{x_2} \lor \overline{x_3}) \land (\overline{x_1} \lor x_3 \lor x_4) \]

**Clause 1**

**Clause 2**

**Clause 3**

**Variable Gadgets**

\[
\begin{array}{ccc}
\overline{x_1} & \overline{x_1} & \overline{x_2} & \overline{x_2} & \overline{x_3} & \overline{x_3} & \overline{x_4} & \overline{x_4} \\
\end{array}
\]

\[2m \text{ nodes}\]

**Clause Gadgets**

\[3/ \text{ nodes}\]

**Clause 1**

**Clause 2**

**Clause 3**
\[
\varphi = (x_1 \lor x_2 \lor x_3) \land (\overline{x_1} \lor \overline{x_2} \lor \overline{x_4}) \land (x_1 \lor x_3 \lor x_4)
\]
First direction in proof:

If $\varphi$ is satisfied, then $G$ contains a vertex cover of size

$$k = m + 2l$$
Example:

\[
\varphi = (x_1 \lor x_2 \lor x_3) \land (\overline{x_1} \lor \overline{x_2} \lor \overline{x_4}) \land (x_1 \lor x_3 \lor x_4)
\]

Satisfying assignment

\[
x_1 = 1 \quad x_2 = 0 \quad x_3 = 0 \quad x_4 = 1
\]

We will show that \( G \) contains a vertex cover of size

\[
k = m + 2l = 4 + 2 \cdot 3 = 10
\]
\[
\varphi = \left( x_1 \lor x_2 \lor x_3 \right) \land \left( \overline{x_1} \lor x_2 \lor \overline{x_4} \right) \land \left( x_1 \lor \overline{x_3} \lor x_4 \right)
\]

\[
x_1 = 1 \quad x_2 = 0 \quad x_3 = 0 \quad x_4 = 1
\]

Put every satisfying literal in the cover
\[ \varphi = (x_1 \lor x_2 \lor x_3) \land (\overline{x_1} \lor x_2 \lor x_4) \land (x_1 \lor \overline{x_3} \lor x_4) \]

\[ x_1 = 1 \quad x_2 = 0 \quad x_3 = 0 \quad x_4 = 1 \]

Select one satisfying literal in each clause gadget and include the remaining literals in the cover.
This is a vertex cover since every edge is adjacent to a chosen node.
Explanation for general case:

Edges in variable gadgets are incident to at least one node in cover
Edges in clause gadgets are incident to at least one node in cover, since two nodes are chosen in a clause gadget.
Every edge connecting variable gadgets and clause gadgets is one of three types:

Type 1

Type 2

Type 3

All adjacent to nodes in cover
Second direction of proof:

If graph \( G \) contains a vertex-cover of size \( k = m + 2l \) then formula \( \varphi \) is satisfiable
Example:
To include “internal’’ edges to gadgets, and satisfy $k = m + 2l$

exactly one literal in each variable gadget is chosen

$\begin{align*}
\text{x}_1 & \quad \text{x}_1 \\
\text{x}_2 & \quad \text{x}_2 \\
\text{x}_3 & \quad \text{x}_3 \\
\text{x}_4 & \quad \text{x}_4
\end{align*}$

$m$ chosen out of $2m$

exactly two nodes in each clause gadget is chosen

$\begin{align*}
\text{x}_1 & \quad \text{x}_1 \\
\text{x}_2 & \quad \text{x}_2 \\
\text{x}_3 & \quad \text{x}_3 \\
\text{x}_4 & \quad \text{x}_4 \\
\text{x}_1 & \quad \text{x}_1 \\
\text{x}_2 & \quad \text{x}_2 \\
\text{x}_3 & \quad \text{x}_3 \\
\text{x}_4 & \quad \text{x}_4
\end{align*}$

$2l$ chosen out of $3l$
For the variable assignment choose the literals in the cover from variable gadgets

\[ x_1 = 1 \quad x_2 = 0 \quad x_3 = 0 \quad x_4 = 1 \]

\[ \varphi = (x_1 \lor x_2 \lor x_3) \land (\overline{x_1} \lor x_2 \lor \overline{x_4}) \land (\overline{x_1} \lor x_3 \lor x_4) \]
\( \varphi = (x_1 \lor x_2 \lor x_3) \land (\overline{x_1} \lor \overline{x_2} \lor x_4) \land (x_1 \lor \overline{x_3} \lor x_4) \)

is satisfied with

\[
\begin{align*}
x_1 &= 1 \\
x_2 &= 0 \\
x_3 &= 0 \\
x_4 &= 1
\end{align*}
\]

since the respective literals satisfy the clauses.
Theorem: HAMILTONIAN-PATH is NP-complete

Proof:

1. HAMILTONIAN-PATH is in NP
   Can be easily proven

2. We will reduce in polynomial time 3CNF-SAT to HAMILTONIAN-PATH (NP-complete)
\[(x_1 \lor x_2) \land (\overline{x_1} \lor x_2)\]

**Gadget for variable** \(x_1\)

- The directions change
\[(x_1 \lor x_2) \land (\overline{x_1} \lor x_2)\]

Gadget for variable \(x_2\)

\[(x_1 \lor x_2)\]

\[(\overline{x_1} \lor x_2)\]
\((x_1 \lor x_2) \land (\overline{x_1} \lor x_2)\)
\((x_1 \lor x_2) \land (\overline{x_1} \lor x_2) = 1\)
\[(x_1 \lor x_2) \land (\overline{x_1} \lor x_2) = 1\]