More NP-complete Problems

Theorem: (proven in previous class)

If: Language A is NP-complete

Language B is in NP

A is polynomial time reducible to B

Then: B is NP-complete

Using the previous theorem, we will prove that 2 problems are NP-complete:

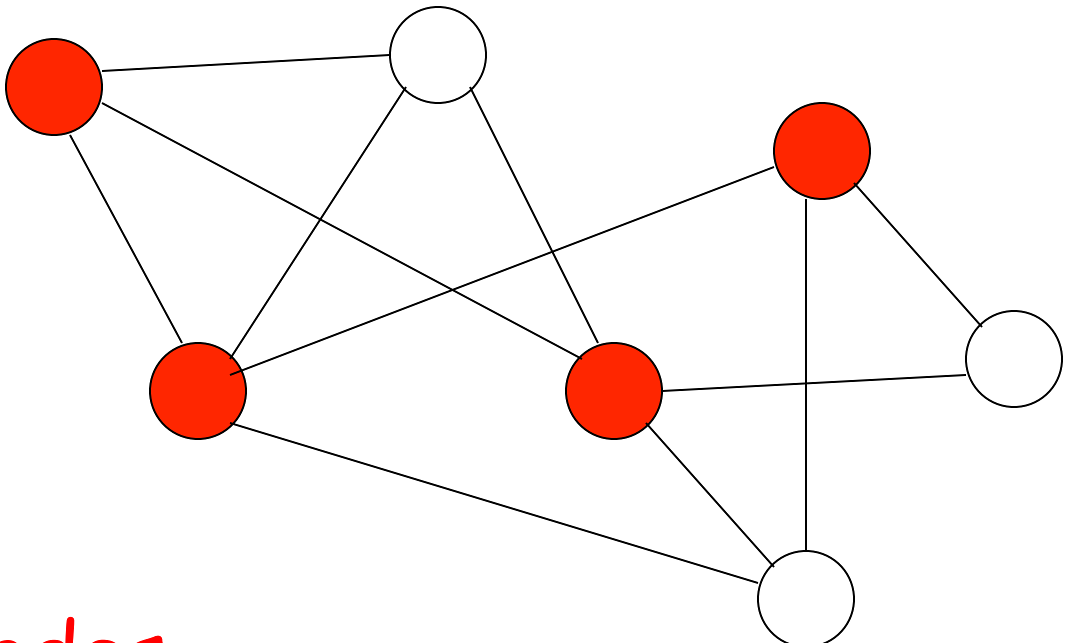
Vertex-Cover

Hamiltonian-Path

Vertex Cover

Vertex cover of a graph
is a subset of nodes S such that every edge
in the graph touches one node in S

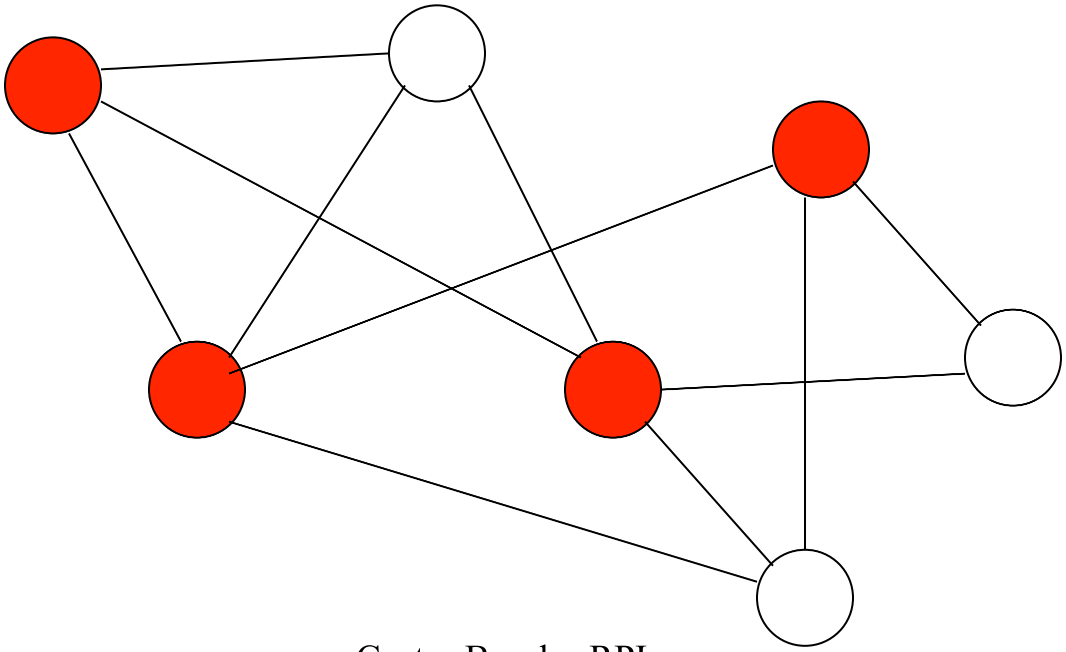
Example:



S = red nodes

Size of vertex-cover is the number of nodes in the cover

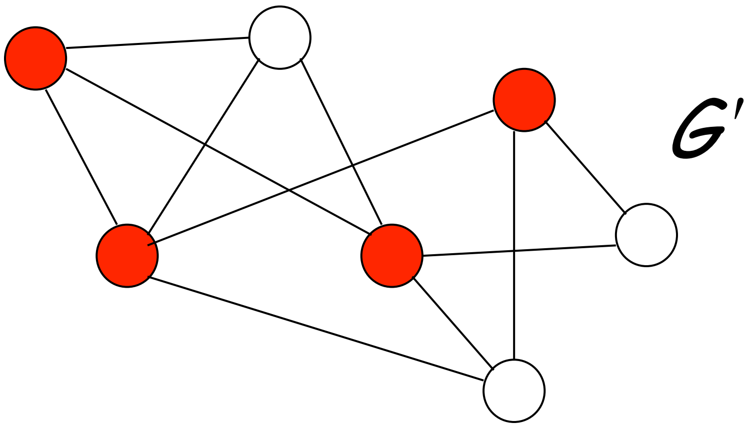
Example: $|S|=4$



Corresponding language:

$\text{VERTEX-COVER} = \{ \langle G, k \rangle :$ graph G contains a vertex cover of size k $\}$

Example:



$\langle G', 4 \rangle \in \text{VERTEX-COVER}$

Theorem: VERTEX-COVER is NP-complete

Proof:

1. VERTEX-COVER is in NP
   Can be easily proven

2. We will reduce in polynomial time
   3CNF-SAT to VERTEX-COVER
   (NP-complete)

Let φ be a 3CNF formula
with m variables
and l clauses

Example:

$$\varphi = (x_1 \vee x_2 \vee x_3) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_4}) \wedge (\overline{x_1} \vee x_3 \vee x_4)$$

Clause 1 Clause 2 Clause 3

$$m = 4$$

$$l = 3$$

Formula φ can be converted to a graph G such that:

φ is satisfied

if and only if

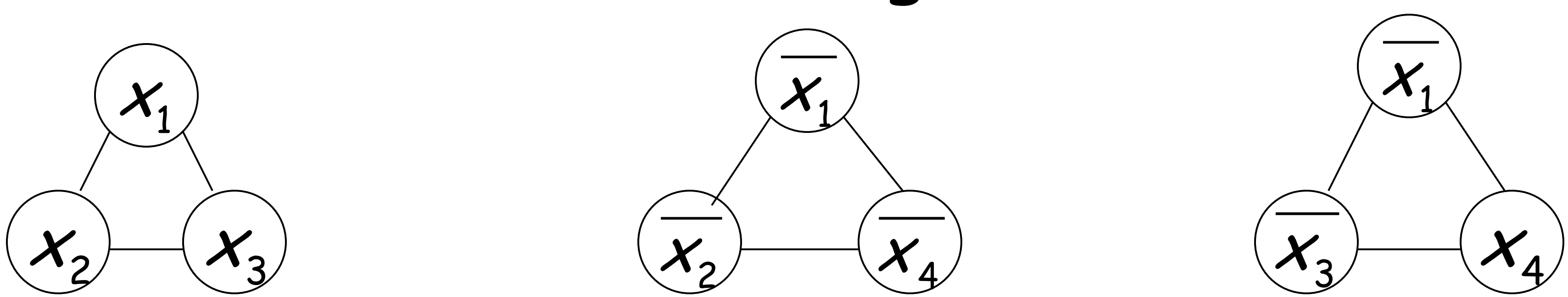
G Contains a vertex cover of size $k = m + 2l$

$$\varphi = (x_1 \vee x_2 \vee x_3) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_4}) \wedge (\overline{x_1} \vee \overline{x_3} \vee x_4)$$

Clause 1 Clause 2 Clause 3

Variable Gadgets $2m$ nodes

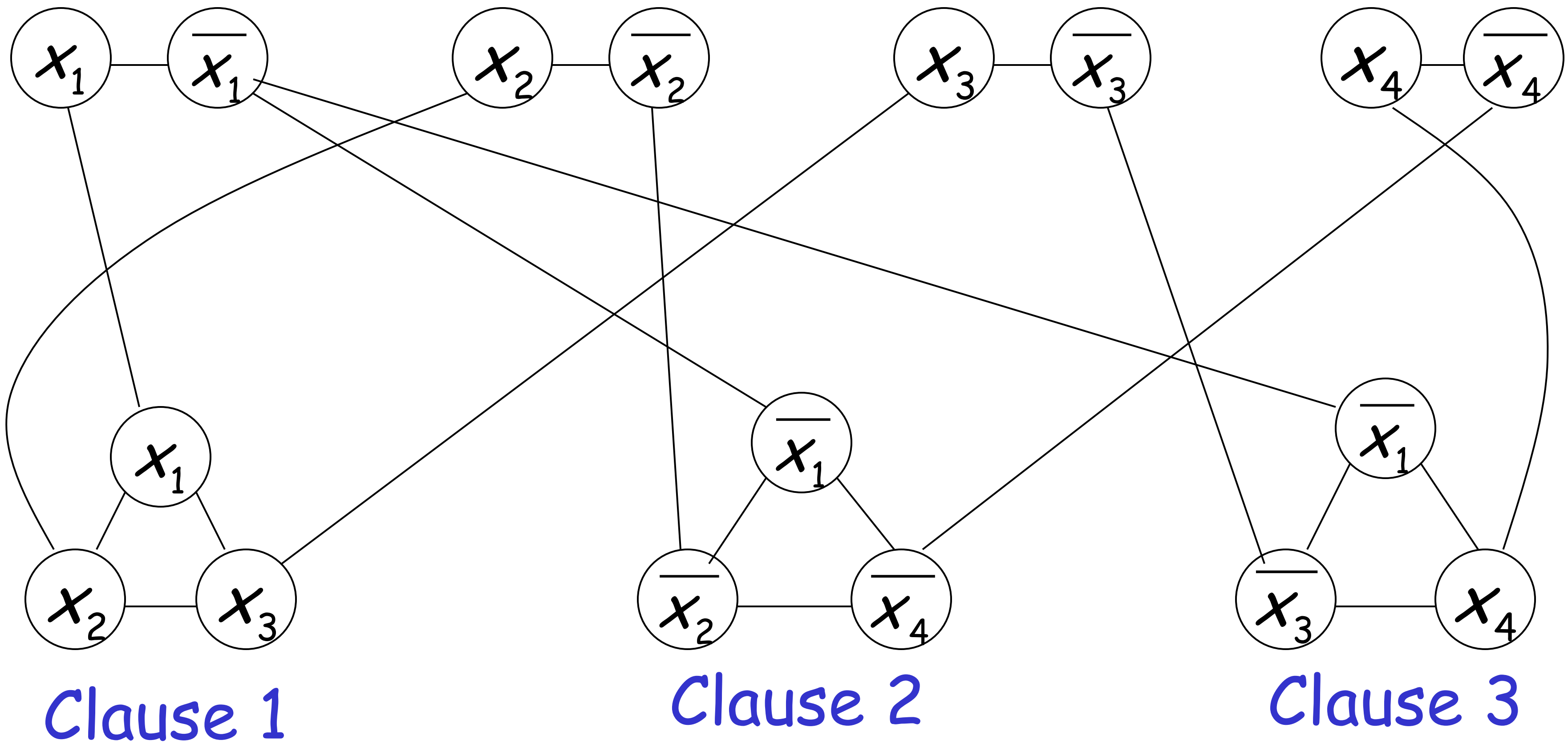
Clause Gadgets 3/ nodes



Clause 1 Clause 2 Clause 3

$$\varphi = (x_1 \vee x_2 \vee x_3) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_4}) \wedge (\overline{x_1} \vee \overline{x_3} \vee x_4)$$

Clause 1 Clause 2 Clause 3



First direction in proof:

If φ is satisfied,
then G contains a vertex cover of size

$$k = m + 2l$$

Example:

$$\varphi = (x_1 \vee x_2 \vee x_3) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_4}) \wedge (\overline{x_1} \vee \overline{x_3} \vee x_4)$$

Satisfying assignment

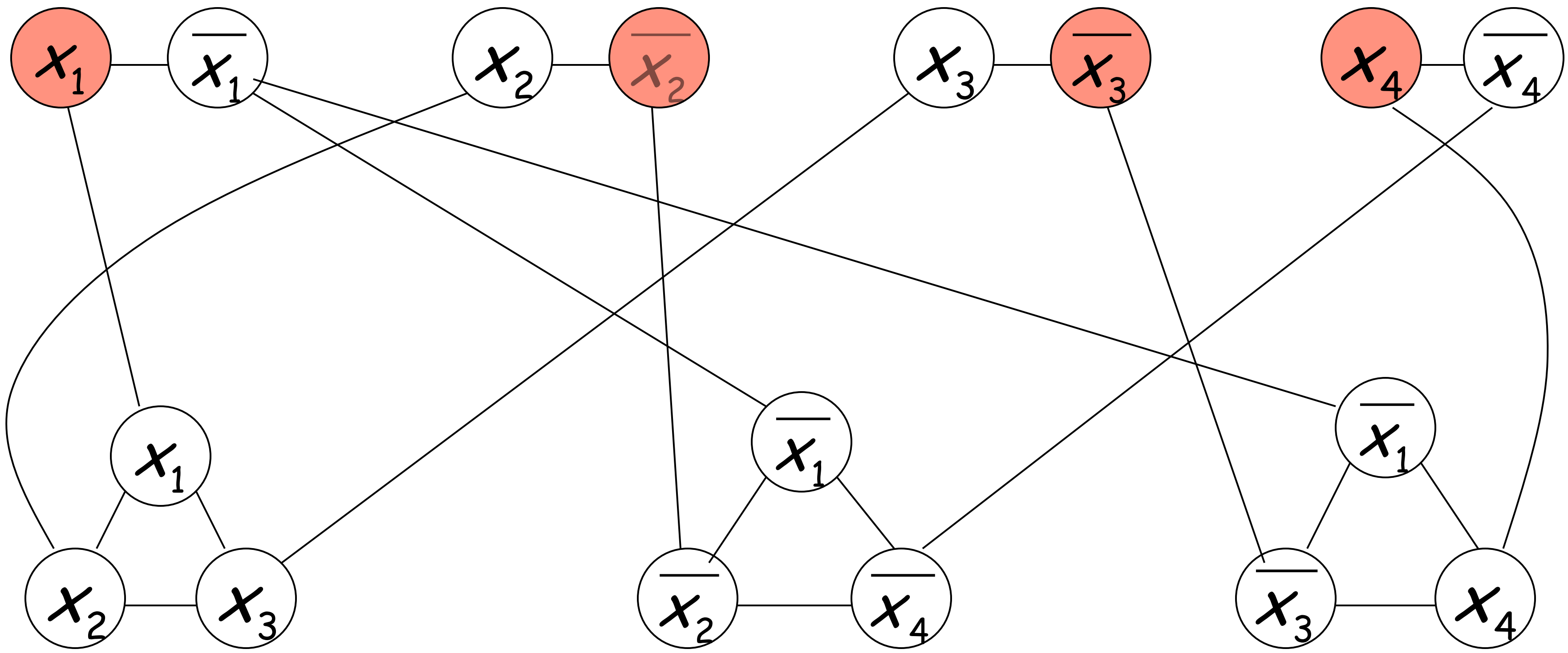
$x_1 = 1 \qquad x_2 = 0 \qquad x_3 = 0 \qquad x_4 = 1$

We will show that G contains a vertex cover of size

$$k = m + 2l = 4 + 2 \cdot 3 = 10$$

$$\varphi = (x_1 \vee x_2 \vee x_3) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_4}) \wedge (\overline{x_1} \vee \overline{x_3} \vee x_4)$$

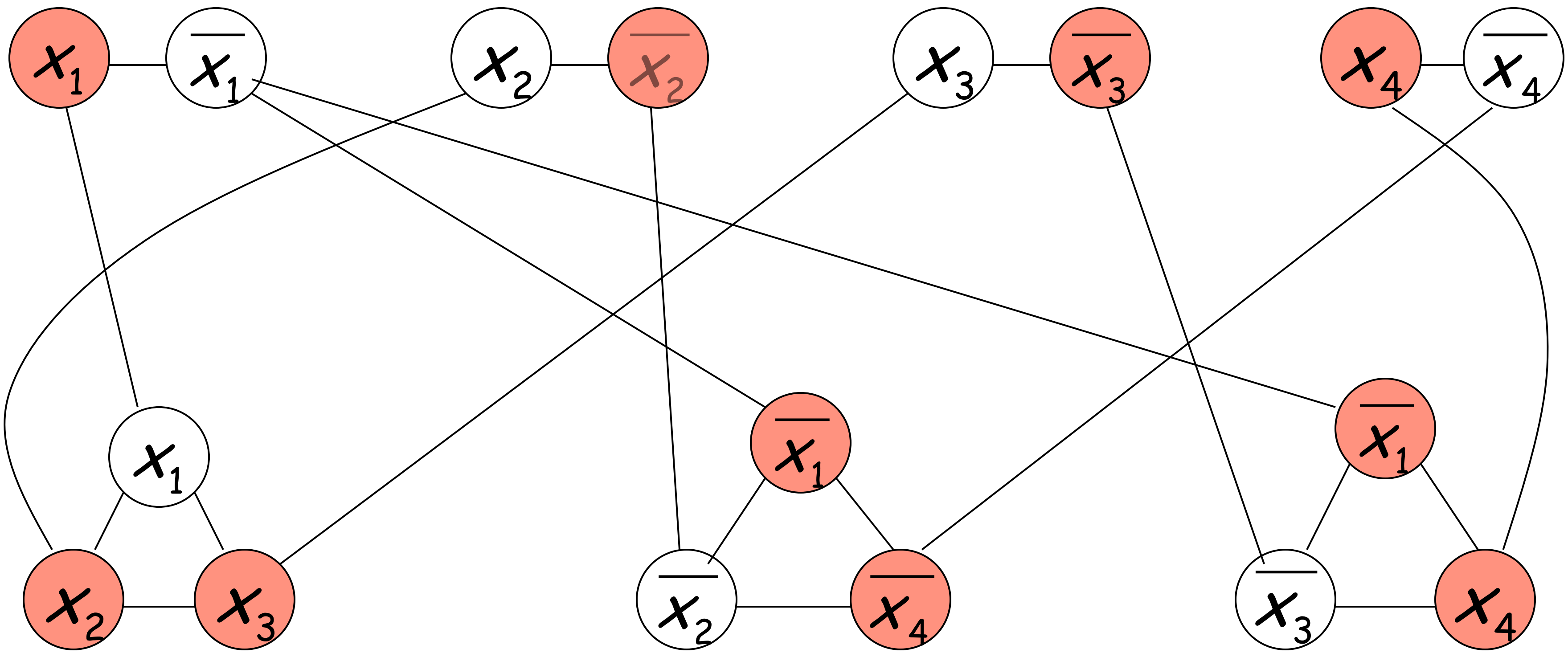
$$x_1 = 1 \qquad x_2 = 0 \qquad x_3 = 0 \qquad x_4 = 1$$



Put every satisfying literal in the cover

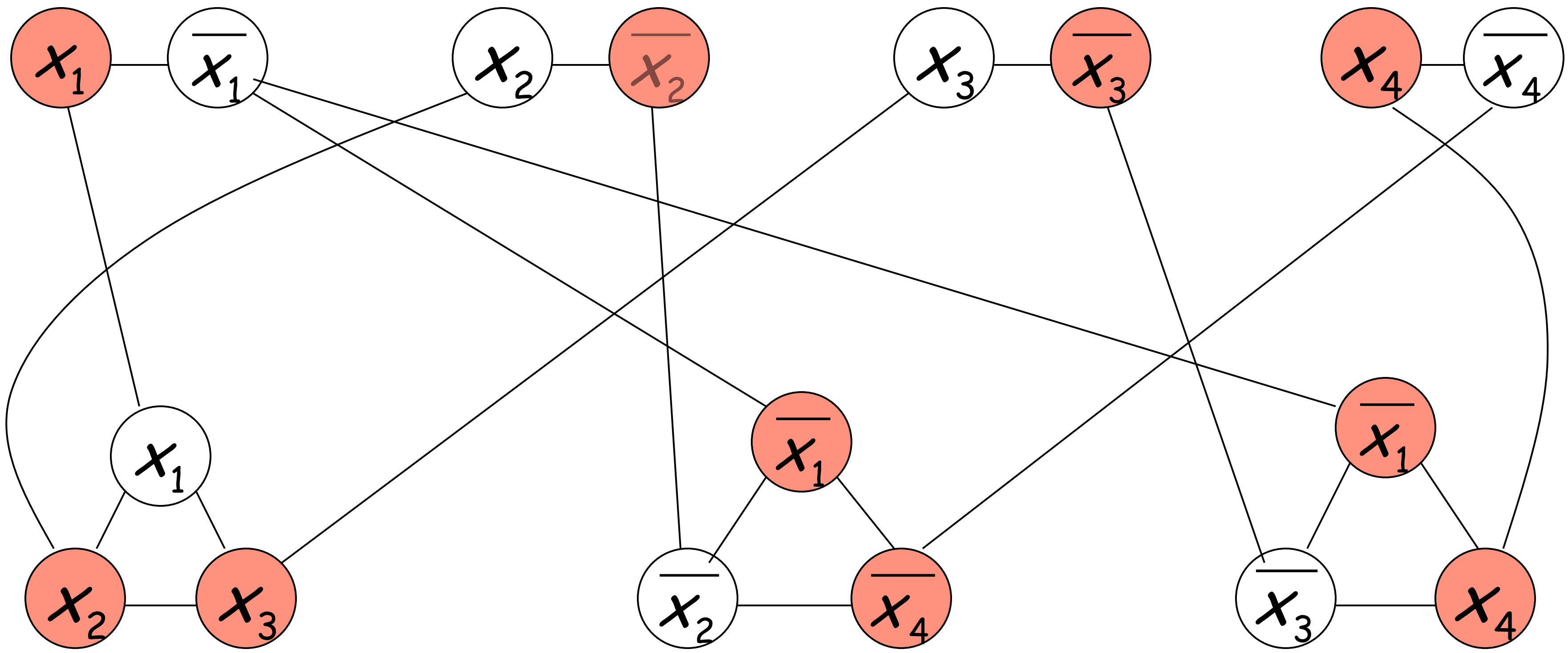
$$\varphi = (x_1 \vee x_2 \vee x_3) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_4}) \wedge (\overline{x_1} \vee \overline{x_3} \vee x_4)$$

$x_1 = 1 \quad x_2 = 0 \quad x_3 = 0 \quad x_4 = 1$



Select one satisfying literal in each clause gadget and include the remaining literals in the cover

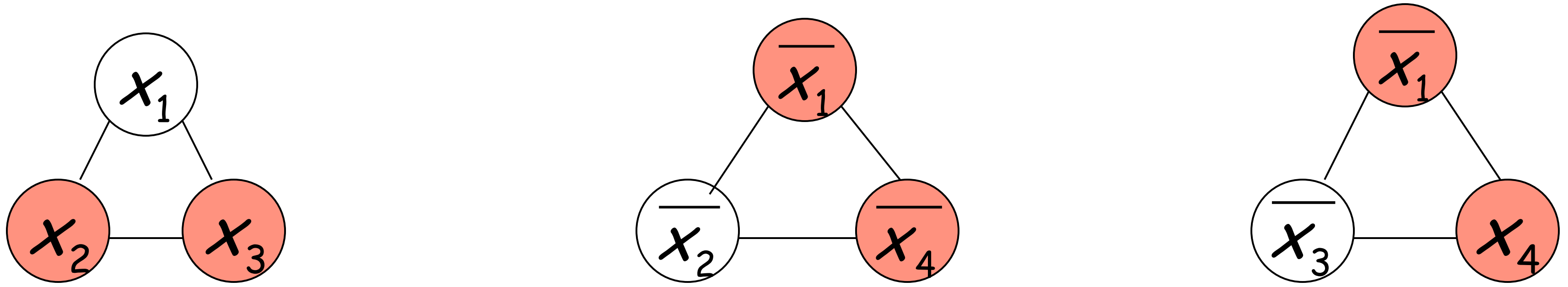
This is a vertex cover since every edge is adjacent to a chosen node



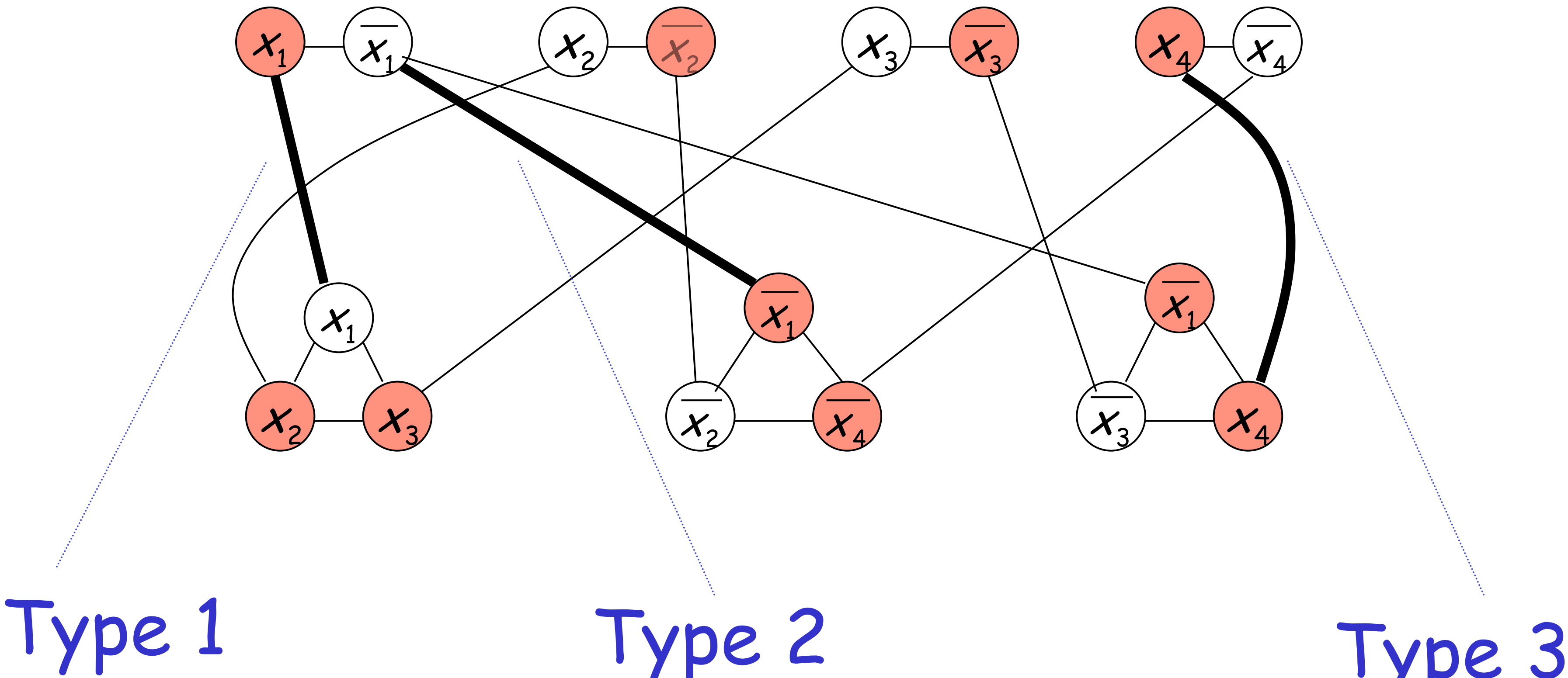
Explanation for general case:

Edges in variable gadgets
are incident to at least one node in cover

Edges in clause gadgets
are incident to at least one node in cover,
since two nodes are chosen in a clause gadget



Every edge connecting variable gadgets and clause gadgets is one of three types:

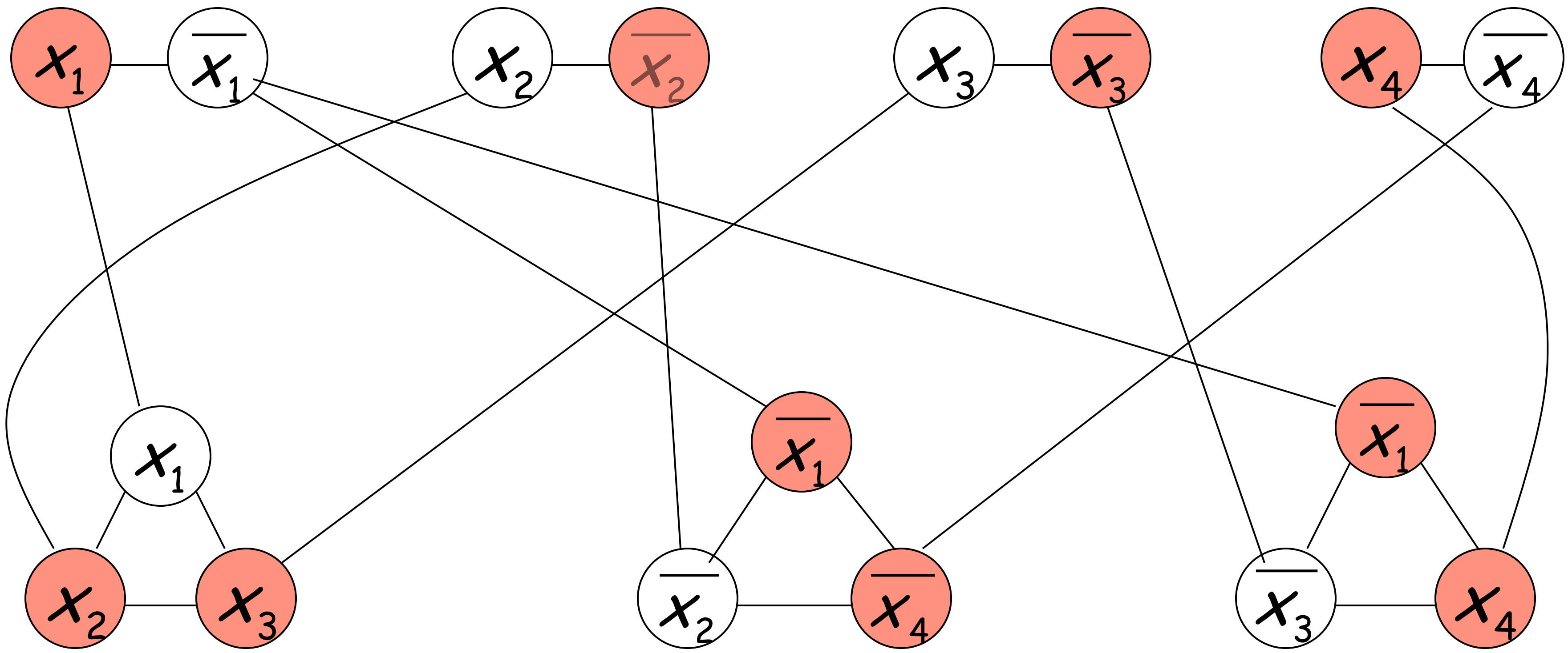


All adjacent to nodes in cover

Second direction of proof:

If graph G contains a vertex-cover
of size $k = m + 2l$
then formula φ is satisfiable

Example:



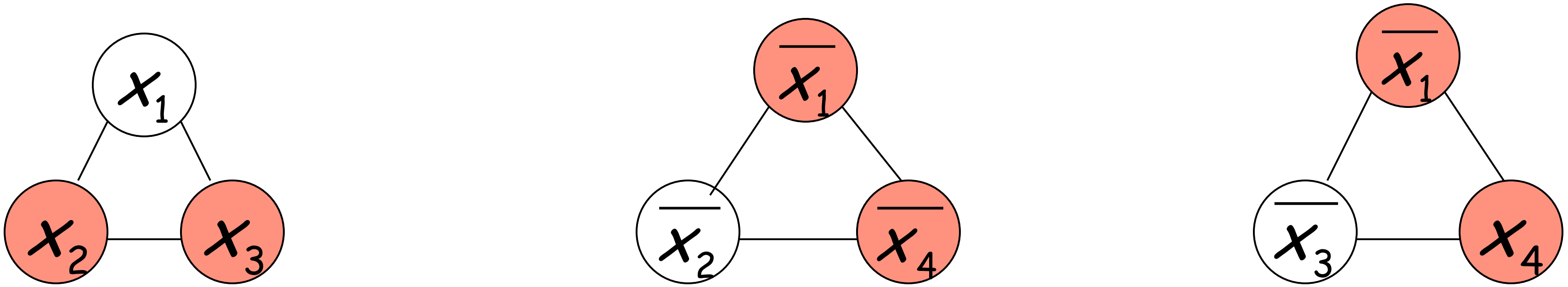
x_1
$\overline{x_1}$
x_2
$\overline{x_2}$
x_3
$\overline{x_3}$
x_4
$\overline{x_4}$
x_1
x_2
x_3
$\overline{x_1}$
$\overline{x_2}$
$\overline{x_4}$
$\overline{x_1}$
$\overline{x_3}$
x_4

To include "internal" edges to gadgets, and satisfy $k = m + 2l$

exactly one literal in each variable gadget is chosen

m chosen out of $2m$

exactly two nodes in each clause gadget is chosen



$2l$ chosen out of $3l$

For the variable assignment choose the literals in the cover from variable gadgets

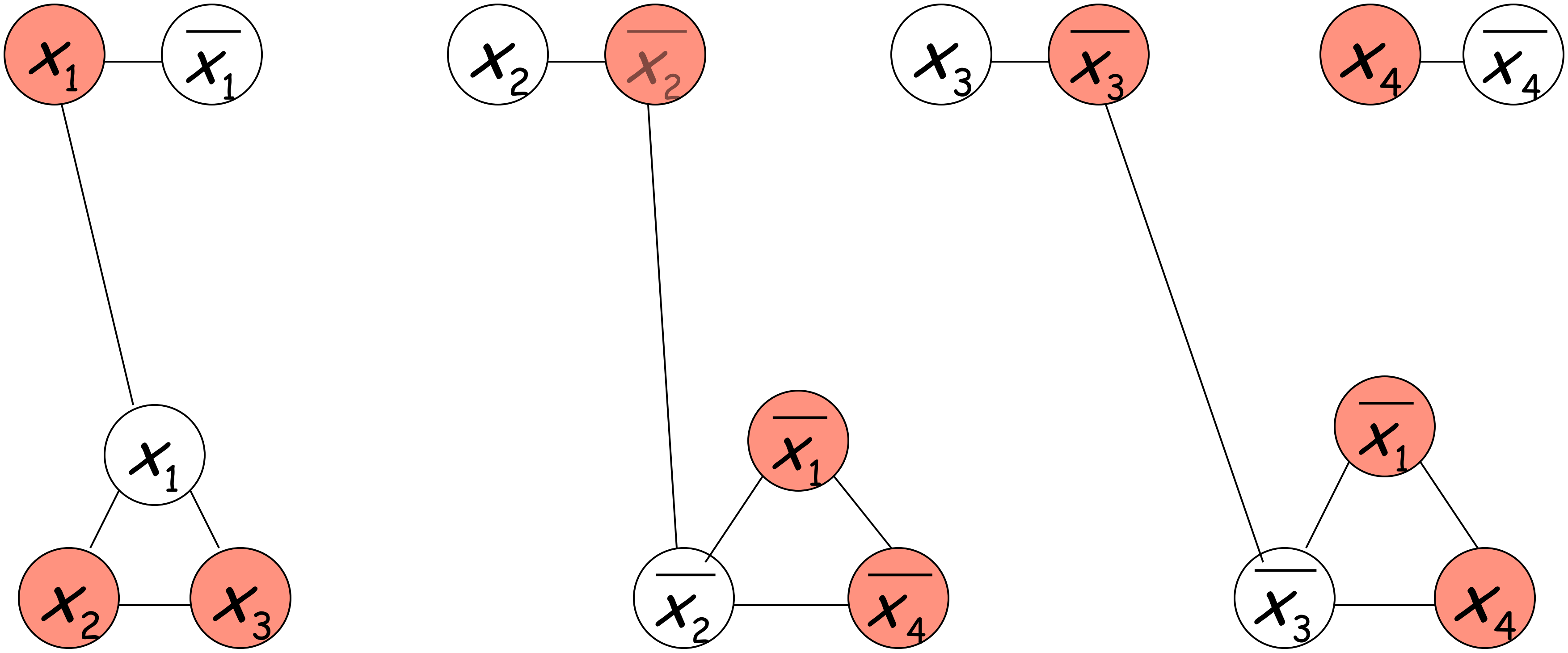
$x_1 = 1$ $\quad x_2 = 0$ $\quad x_3 = 0$ $\quad x_4 = 1$

$$\varphi = (x_1 \vee x_2 \vee x_3) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_4}) \wedge (\overline{x_1} \vee \overline{x_3} \vee x_4)$$

$$\varphi = (x_1 \vee x_2 \vee x_3) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_4}) \wedge (\overline{x_1} \vee \overline{x_3} \vee x_4)$$

is satisfied with

$x_1 = 1$ $x_2 = 0$ $x_3 = 0$ $x_4 = 1$



since the respective literals satisfy the clauses

Theorem: HAMILTONIAN-PATH is NP-complete

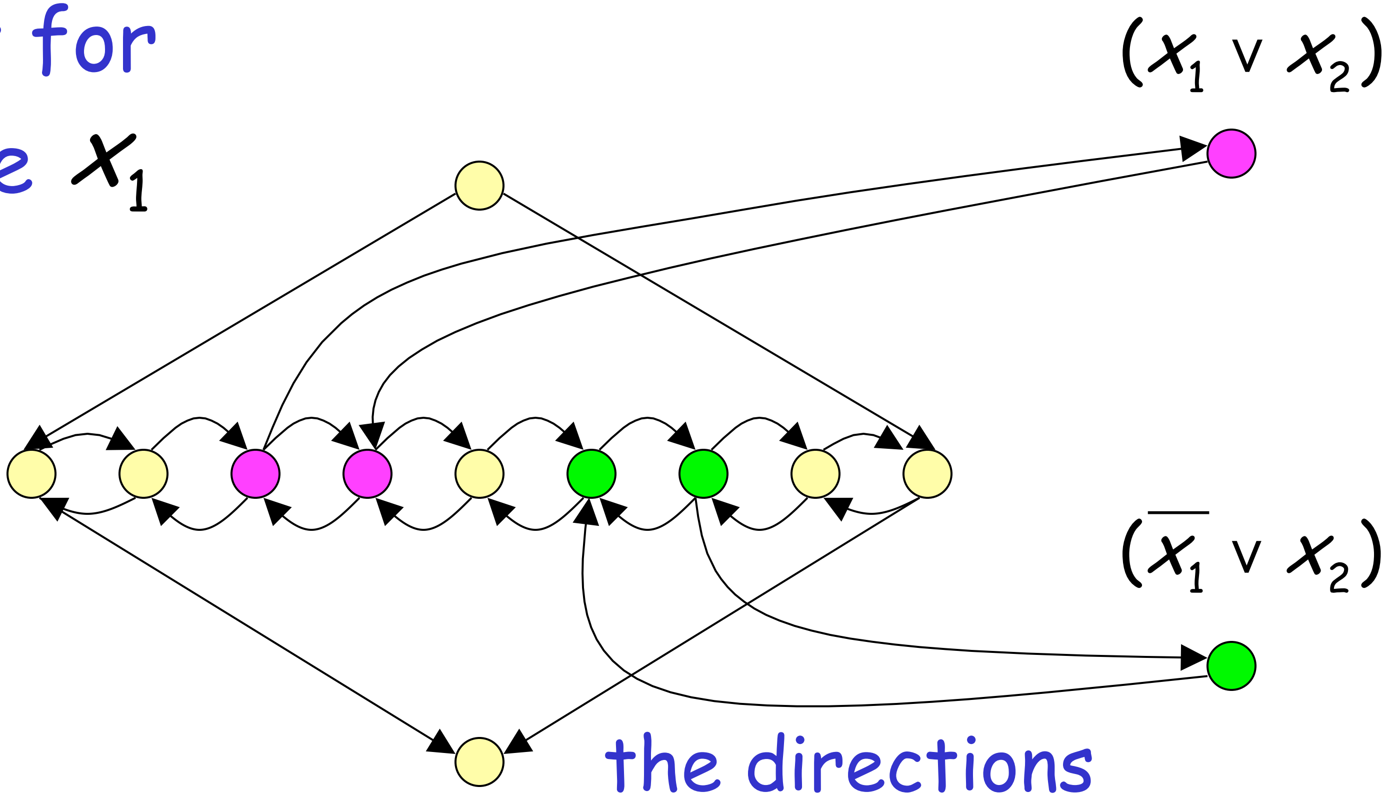
Proof:

1. HAMILTONIAN-PATH is in NP
   Can be easily proven

2. We will reduce in polynomial time 3CNF-SAT to HAMILTONIAN-PATH (NP-complete)

$$(x_1 \vee x_2) \wedge (\overline{x_1} \vee x_2)$$

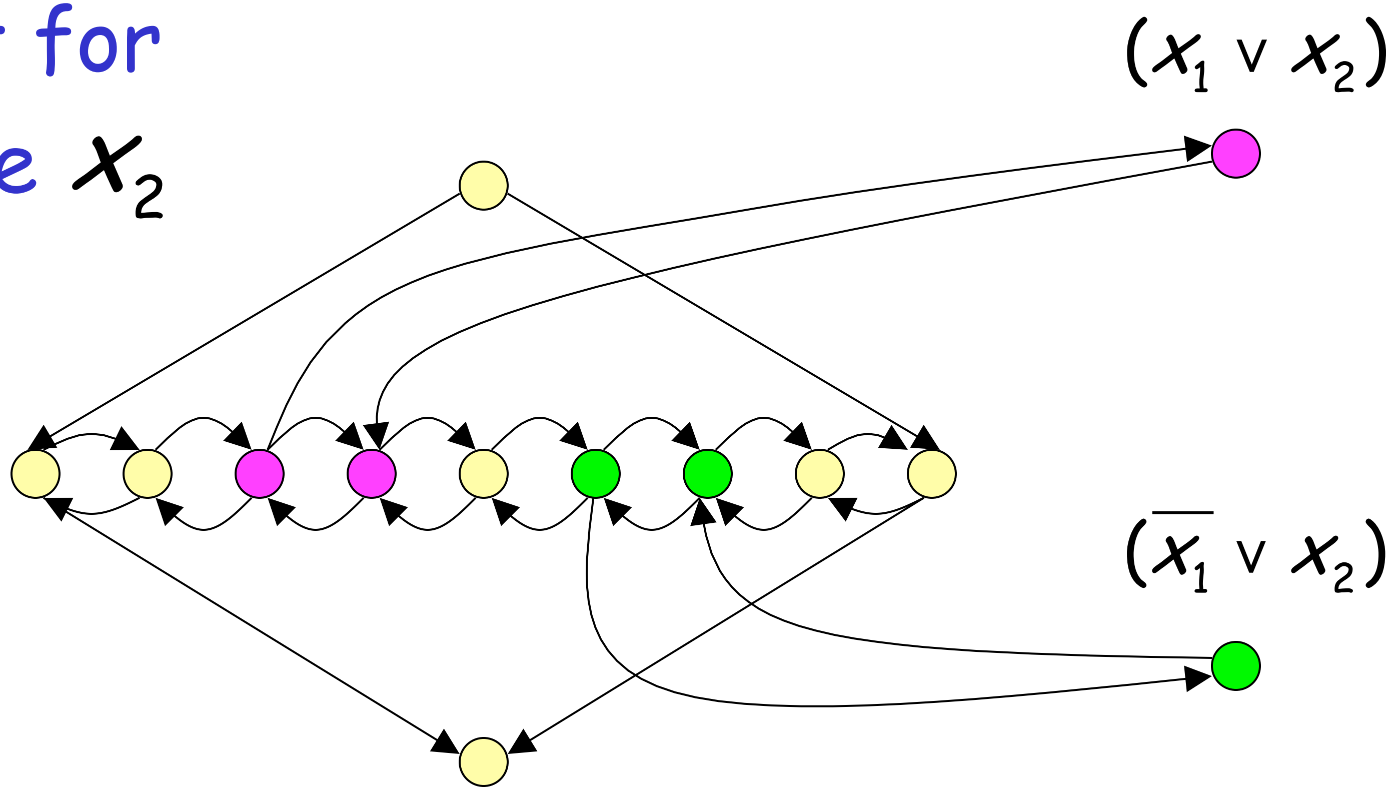
Gadget for variable x_1



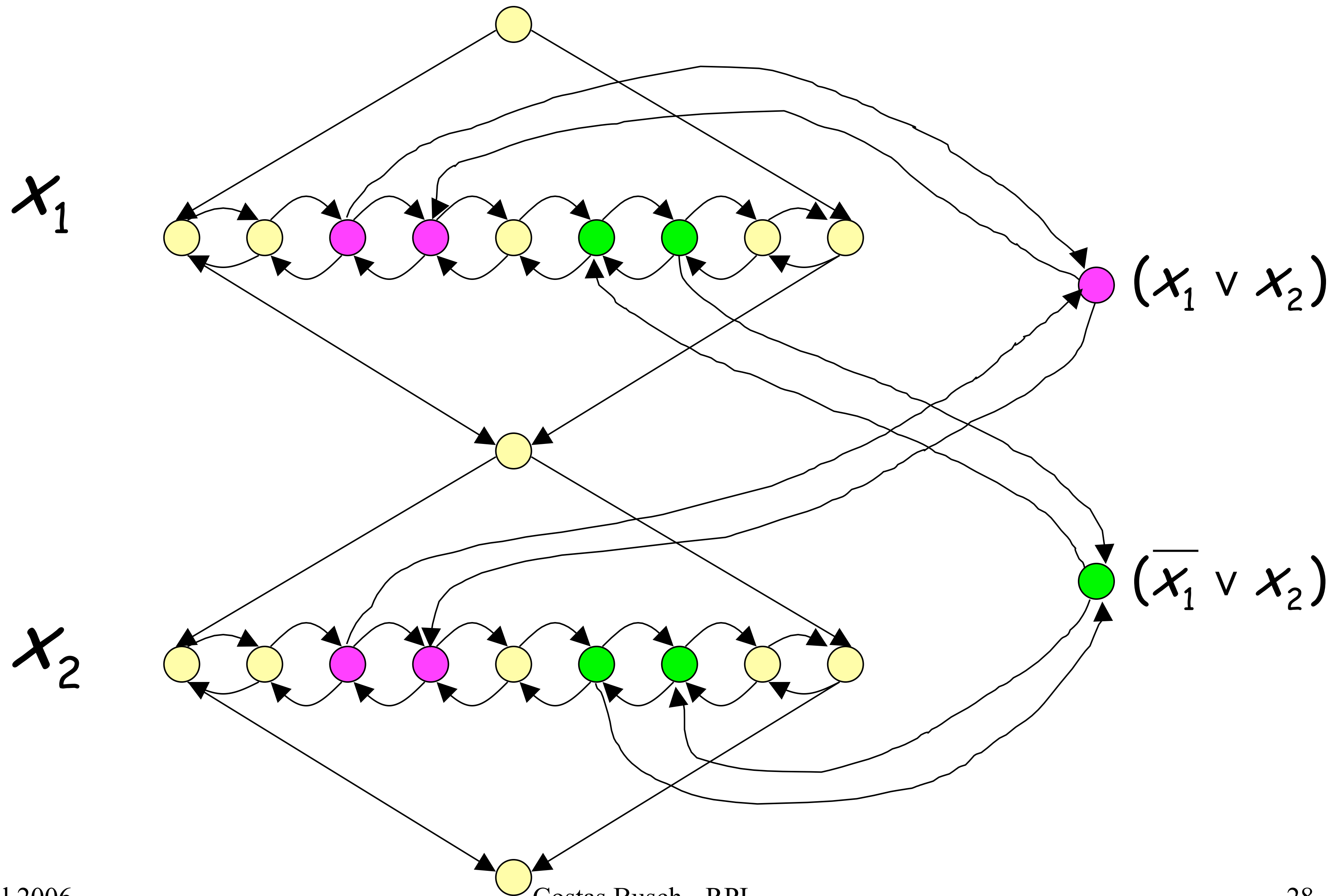
the directions change

$$(x_1 \lor x_2) \land (\overline{x_1} \lor x_2)$$

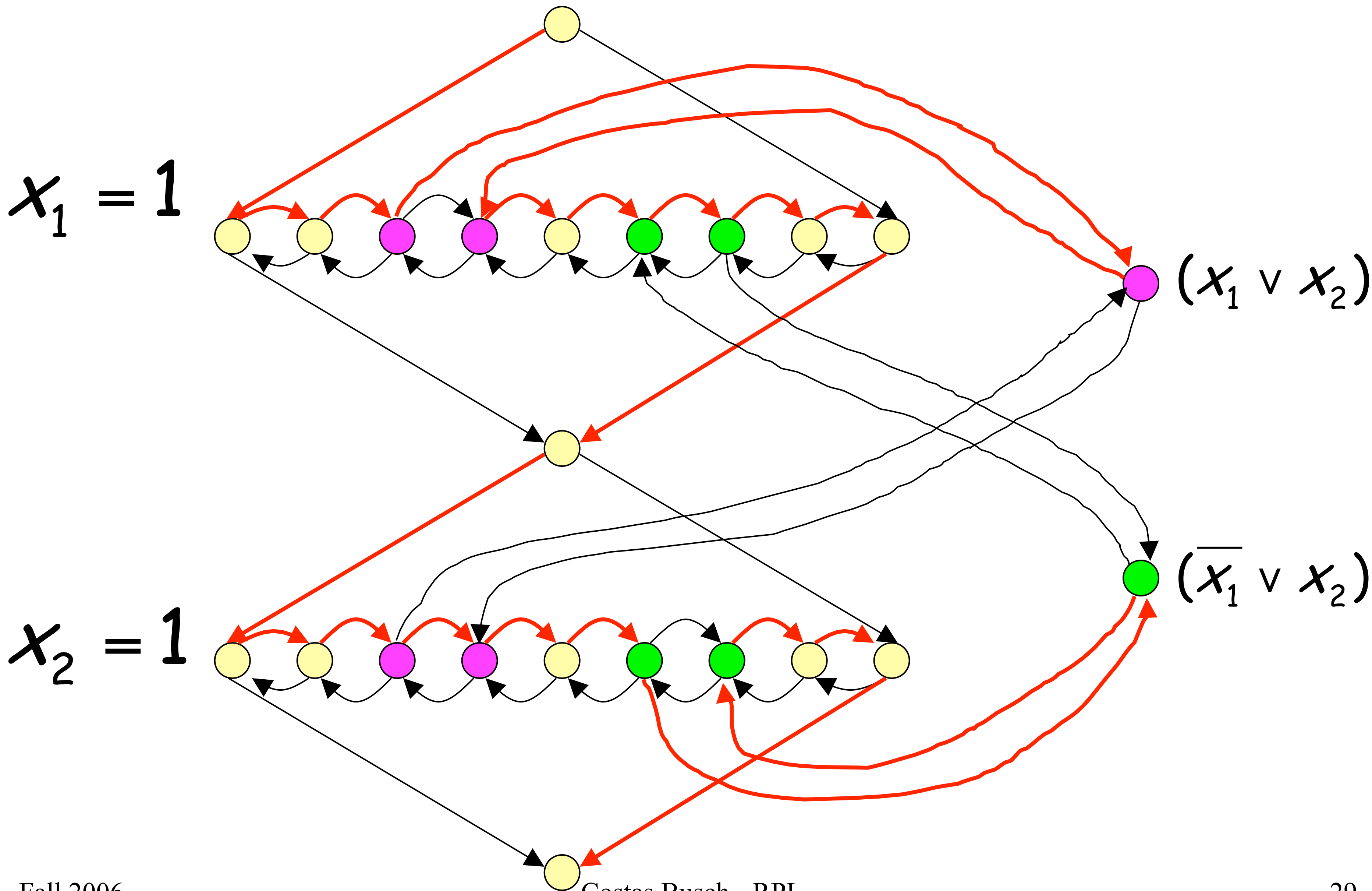
Gadget for variable x_2



$$(x_1 \vee x_2) \wedge (\overline{x_1} \vee x_2)$$



$$(x_1 \vee x_2) \wedge (\overline{x_1} \vee x_2) = 1$$



$$(x_1 \vee x_2) \wedge (\overline{x_1} \vee x_2) = 1$$