Introduction to Finite Automata

Languages

Deterministic Finite Automata

Representations of Automata

Alphabets

- An *alphabet* is any finite set of symbols.
- Examples: ASCII, Unicode, {0,1} (*binary alphabet*), {a,b,c}.

Strings

- The set of *strings* over an alphabet Σ is the set of lists, each element of which is a member of Σ.
  - Strings shown with no commas, e.g., abc.
- Σ^* denotes this set of strings.
- ϵ stands for the *empty string* (string of length 0).

Example: Strings

- $\{0,1\}^* = \{\epsilon, 0, 1, 00, 01, 10, 11, 000, 001, \ldots\}$
- Subtlety: 0 as a string, 0 as a symbol look the same.
  - Context determines the type.

Languages

- A *language* is a subset of Σ^* for some alphabet Σ.
- Example: The set of strings of 0's and 1's with no two consecutive 1's.
- $L = \{\epsilon, 0, 1, 00, 01, 10, 000, 001, 010, 100, 101, 0000, 0001, 0010, 0100, 0101, 1000, 1001, 1010, \ldots\}$

Deterministic Finite Automata

- A formalism for defining languages, consisting of:
  1. A finite set of *states* (Q, typically).
  2. An *input alphabet* (Σ, typically).
  3. A *transition function* (δ, typically).
  4. A *start state* (q_0, in Q, typically).
  5. A set of *final states* ($F \subseteq Q$, typically).
     - "Final" and "accepting" are synonyms.

The Transition Function

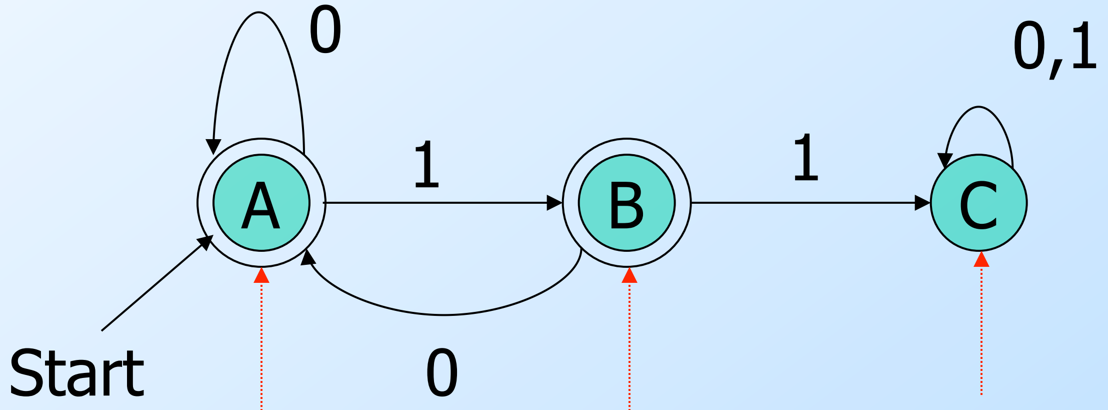
- Takes two arguments: a state and an input symbol.
- $\delta(q, a)$ = the state that the DFA goes to when it is in state *q* and input *a* is received.

Graph Representation of DFA’s

- Nodes = states.
- Arcs represent transition function.
  - Arc from state p to state q labeled by all those input symbols that have transitions from p to q.
- Arrow labeled “Start” to the start state.
- Final states indicated by double circles.

Example: Graph of a DFA

Accepts all strings without two consecutive 1's.



Previous string OK, does not end in 1.

Previous String OK, ends in a single 1.

Consecutive 1's have been seen.

Alternative Representation: Transition Table

Final states starred

Columns = input symbols

Arrow for start state

	0	1
→ * A	A	B
* B	A	C
C	C	C

Rows = states

Extended Transition Function

- We describe the effect of a string of inputs on a DFA by extending δ to a state and a string.
- Induction on length of string.
- Basis: $\delta(q, \epsilon) = q$
- Induction: $\delta(q,wa) = \delta(\delta(q,w),a)$
  - w is a string; a is an input symbol.

Extended δ: Intuition

- Convention:
  - … w, x, y, x are strings.
  - a, b, c,… are single symbols.
- Extended δ is computed for state q and inputs $a_1a_2…a_n$ by following a path in the transition graph, starting at q and selecting the arcs with labels a_1, a_2,…,a_n in turn.

Example: Extended Delta

	0	1
A	A	B
B	A	C
C	C	C

$$\delta(B,011) = \delta(\delta(B,01),1) = \delta(\delta(\delta(B,0),1),1) =$$

$$\delta(\delta(A,1),1) = \delta(B,1) = C$$

Delta-hat

- Some people denote the extended δ with a "hat" to distinguish it from δ itself.
- Not needed, because both agree when the string is a single symbol.
- $\hat{\delta}(q, a) = \delta(\hat{\delta}(q, \epsilon), a) = \delta(q, a)$

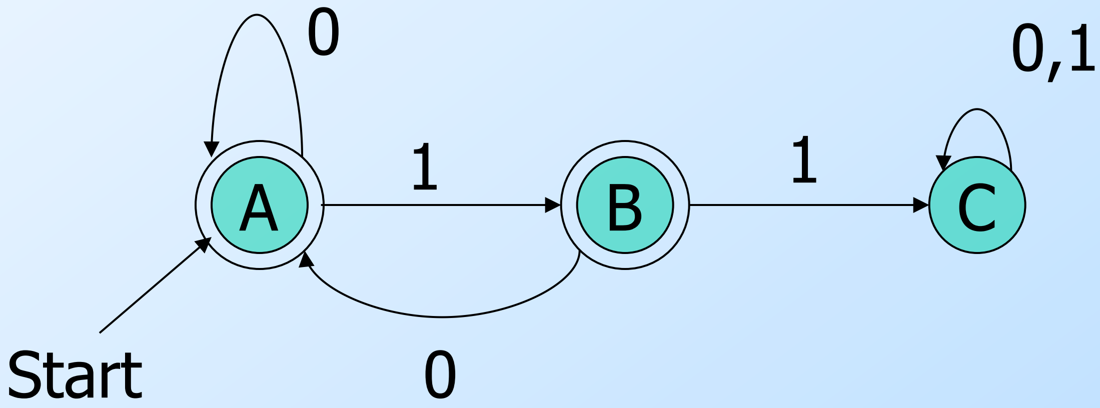
Extended deltas

Language of a DFA

- Automata of all kinds define languages.
- If A is an automaton, L(A) is its language.
- For a DFA A, L(A) is the set of strings labeling paths from the start state to a final state.
- Formally: L(A) = the set of strings w such that $\delta(q_0, w)$ is in F.

Example: String in a Language

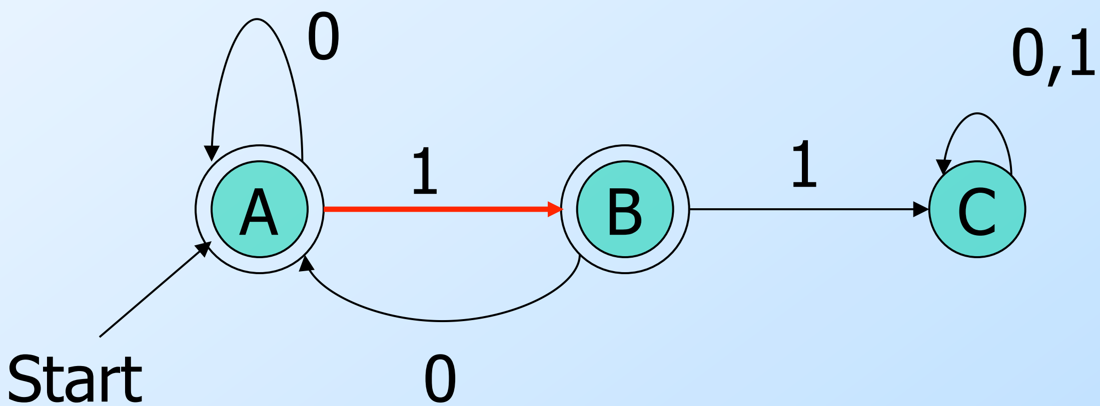
String 101 is in the language of the DFA below. Start at A.



Example: String in a Language

String 101 is in the language of the DFA below.

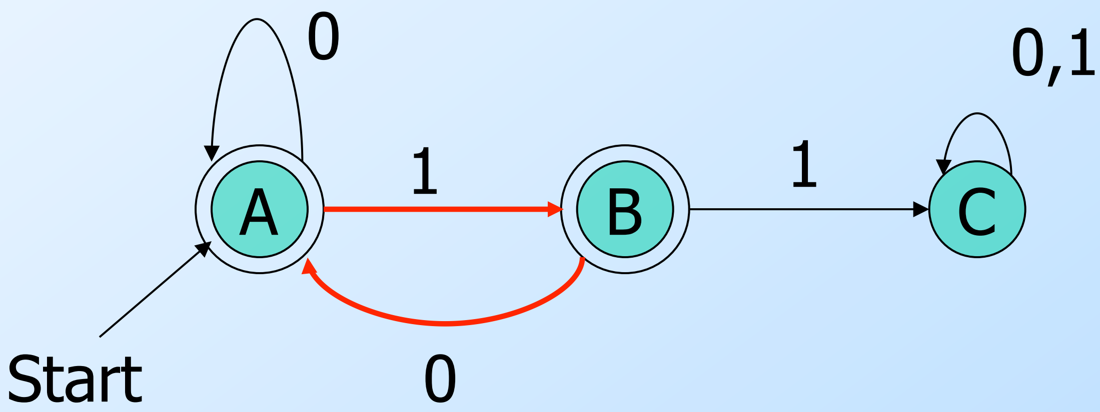
Follow arc labeled 1.



Example: String in a Language

String 101 is in the language of the DFA below.

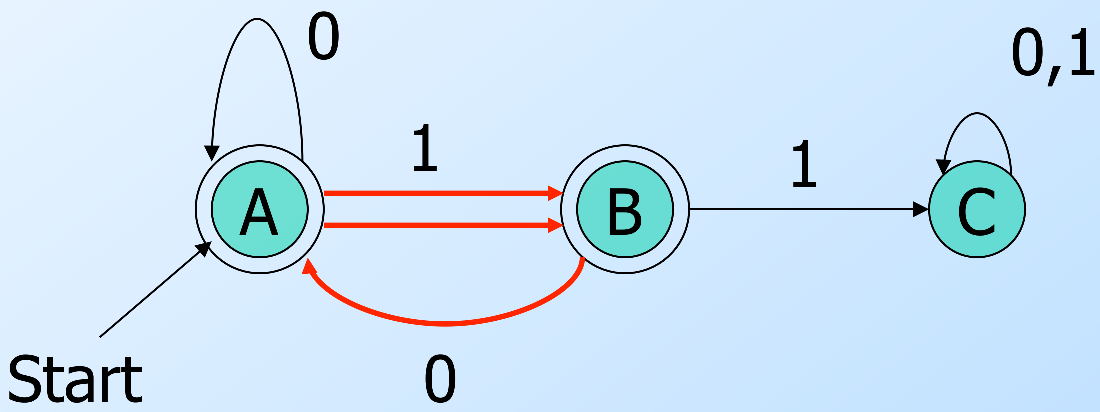
Then arc labeled 0 from current state B.



Example: String in a Language

String 101 is in the language of the DFA below.

Finally arc labeled 1 from current state A. Result is an accepting state, so 101 is in the language.



Example – Concluded

- The language of our example DFA is:

{w | w is in {0,1}* and w does not have two consecutive 1's}

Such that…

These conditions about w are true.

Read a *set former* as "The set of strings w…

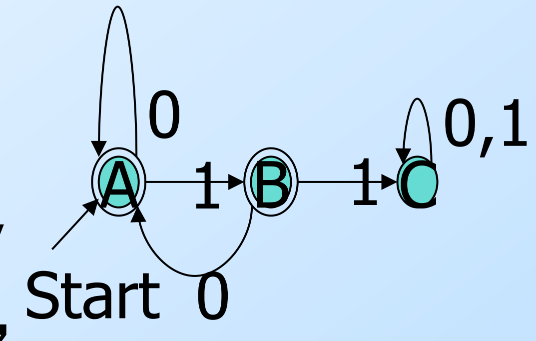
Proofs of Set Equivalence

- Often, we need to prove that two descriptions of sets are in fact the same set.
- Here, one set is "the language of this DFA," and the other is "the set of strings of 0's and 1's with no consecutive 1's."

Proofs – (2)

- In general, to prove S=T, we need to prove two parts: $S \subseteq T$ and $T \subseteq S$. That is:
  1. If w is in S, then w is in T.
  2. If w is in T, then w is in S.
- As an example, let S = the language of our running DFA, and T = "no consecutive 1's."

Part 1: $S \subseteq T$



- To prove: if w is accepted by then w has no consecutive 1's.
- Proof is an induction on length of w.
- Important trick: Expand the inductive hypothesis to be more detailed than you need.

The Inductive Hypothesis

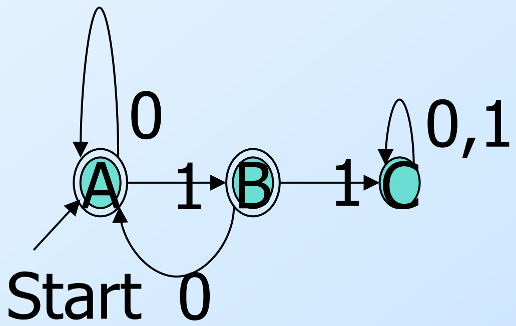
1. If $\delta(A, w) = A$, then w has no consecutive 1's and does not end in 1.
2. If $\delta(A, w) = B$, then w has no consecutive 1's and ends in a single 1.

- Basis: $|w| = 0$; i.e., $w = \epsilon$.
  - (1) holds since ϵ has no 1's at all.
  - (2) holds *vacuously*, since $\delta(A, \epsilon)$ is not B.

"length of"

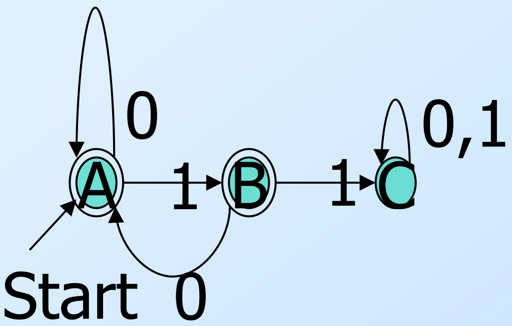
Important concept:
If the "if" part of "if..then" is false, the statement is true.

Inductive Step



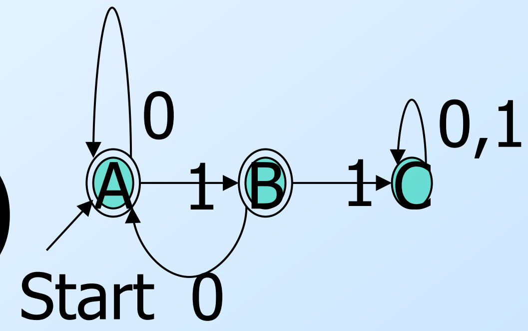
- Assume (1) and (2) are true for strings shorter than w, where |w| is at least 1.
- Because w is not empty, we can write w = xa, where *a* is the last symbol of w, and x is the string that precedes.
- IH is true for x.

Inductive Step – (2)



- Need to prove (1) and (2) for w = xa.
- (1) for w is: If δ(A, w) = A, then w has no consecutive 1's and does not end in 1.
- Since δ(A, w) = A, δ(A, x) must be A or B, and *a* must be 0 (look at the DFA).
- By the IH, x has no 11's.
- Thus, w has no 11's and does not end in 1.

Inductive Step – (3)

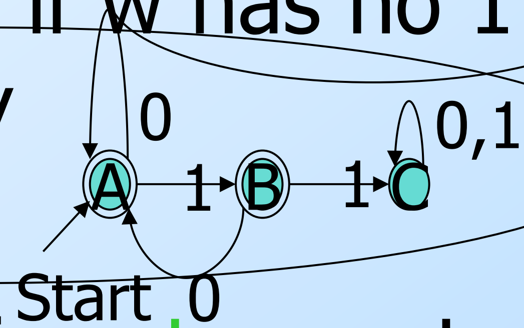


- Now, prove (2) for w = xa: If δ(A, w) = B, then w has no 11's and ends in 1.
- Since δ(A, w) = B, δ(A, x) must be A, and *a* must be 1 (look at the DFA).
- By the IH, x has no 11's and does not end in 1.
- Thus, w has no 11's and ends in 1.

Part 2: $T \subseteq S$

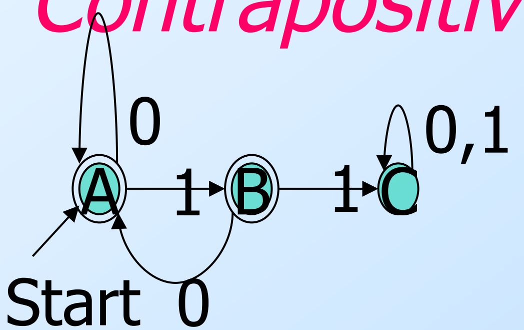
X

- Now, we must prove: if w has no 11's, then w is accepted by



Y

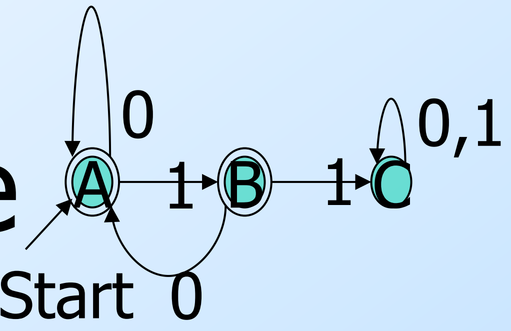
- *Contrapositive* : If w is not accepted by



then w has 11.

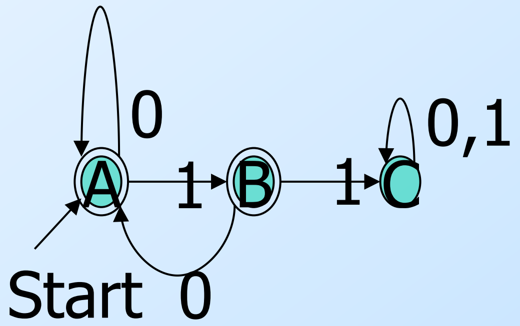
Key idea: contrapositive of "if X then Y" is the equivalent statement "if not Y then not X."

Using the Contrapositive



- Every w gets the DFA to exactly one state.
  - Simple inductive proof based on:
    - Every state has exactly one transition on 1, one transition on 0.
- The only way w is not accepted is if it gets to C.

Using the Contrapositive – (2)



- The only way to get to C [formally: δ(A,w) = C] is if w = x1y, x gets to B, and y is the tail of w that follows what gets to C for the first time.
- If δ(A,x) = B then surely x = z1 for some z.
- Thus, w = z11y and has 11.

Regular Languages

- A language L is ***regular*** if it is the language accepted by some DFA.
  - Note: the DFA must accept only the strings in L, no others.
- Some languages are not regular.
  - Intuitively, regular languages "cannot count" to arbitrarily high integers.

Example: A Nonregular Language

$L_1 = \{0^n1^n \mid n \geq 1\}$

- Note: a^i is conventional for i a's.
  - Thus, $0^4 = 0000$, e.g.
- Read: "The set of strings consisting of n 0's followed by n 1's, such that n is at least 1.
- Thus, $L_1 = \{01, 0011, 000111, \ldots\}$

Another Example

L_2 = {w | w in {(,)}* and w is *balanced* }

- Note: alphabet consists of the parenthesis symbols '(' and ')'.
- Balanced parens are those that can appear in an arithmetic expression.
  - E.g.: (), ()(), (()), (()()),…

But Many Languages are Regular

- Regular Languages can be described in many ways, e.g., regular expressions.
- They appear in many contexts and have many useful properties.
- Example: the strings that represent floating point numbers in your favorite language is a regular language.

Example: A Regular Language

L_3 = { w | w in {0,1}* and w, viewed as a binary integer is divisible by 23}

- The DFA:
  - 23 states, named 0, 1,…,22.
  - Correspond to the 23 remainders of an integer divided by 23.
  - Start and only final state is 0.

Transitions of the DFA for L_3

- If string w represents integer i, then assume $\delta(0, w) = i\%23$.
- Then w0 represents integer 2i, so we want $\delta(i\%23, 0) = (2i)\%23$.
- Similarly: w1 represents 2i+1, so we want $\delta(i\%23, 1) = (2i+1)\%23$.
- Example: $\delta(15,0) = 30\%23 = 7$; $\delta(11,1) = 23\%23 = 0$.

Key idea: design a DFA by figuring out what each state needs to remember about the past.

Another Example

L_4 = { w | w in {0,1}* and w, viewed as the reverse of a binary integer is divisible by 23}

- Example: 01110100 is in L_4, because its reverse, 00101110 is 46 in binary.
- Hard to construct the DFA.
- But theorem says the reverse of a regular language is also regular.