More Undecidable Problems

Rice's Theorem

Post's Correspondence Problem

Some Real Problems

Properties of Languages

- Any set of languages is a *property* of languages.
- Example: The infiniteness property is the set of infinite languages.

Properties of Langauges – (2)

- As always, languages must be defined by some descriptive device.
- The most general device we know is the TM.
- Thus, we shall think of a property as a problem about Turing machines.
- Let L_P be the set of binary TM codes for TM's M such that L(M) has property P.

Trivial Properties

- There are two (*trivial*) properties P for which L_P is decidable.
  1. The *always-false property*, which contains no RE languages.
  2. The *always-true property*, which contains every RE language.
- Rice's Theorem: For every other property P, L_P is undecidable.

Plan for Proof of Rice's Theorem

1. Lemma needed: recursive languages are closed under complementation.
2. We need the technique known as *reduction*, where an algorithm converts instances of one problem to instances of another.
3. Then, we can prove the theorem.

Closure of Recursive Languages Under Complementation

- If L is a language with alphabet Σ^*, then the *complement* of L is Σ^* - L.
  - Denote the complement of L by L^c.
- Lemma: If L is recursive, so is L^c.
- Proof: Let L = L(M) for a TM M.
- Construct M' for L^c.
- M' has one final state, the new state f.

Proof – Concluded

- M′ simulates M.
- But if M enters an accepting state, M′ halts without accepting.
- If M halts without accepting, M′ instead has a move taking it to state f.
- In state f, M′ halts.

Reductions

- A *reduction* from language L to language L' is an algorithm (TM that always halts) that takes a string w and converts it to a string x, with the property that:

  x is in L' if and only if w is in L.

TM's as *Transducers*

- We have regarded TM's as acceptors of strings.
- But we could just as well visualize TM's as having an *output tape*, where a string is written prior to the TM halting.
- Such a TM translates its input to its output.

Reductions – (2)

- If we reduce L to L′, and L′ is decidable, then the algorithm for L′ + the algorithm of the reduction shows that L is also decidable.
- Used in the contrapositive: If we know L is not decidable, then L′ cannot be decidable.

Reductions – Aside

- This form of reduction is not the most general.
- Example: We “reduced” L_d to L_u, but in doing so we had to complement answers.
- More in NP-completeness discussion on Karp vs. Cook reductions.

Proof of Rice's Theorem

- We shall show that for every nontrivial property P of the RE languages, L_P is undecidable.
- We show how to reduce L_u to L_P.
- Since we know L_u is undecidable, it follows that L_P is also undecidable.

The Reduction

- Our reduction algorithm must take M and w and produce a TM M'.
- L(M') has property P if and only if M accepts w.
- M' has two tapes, used for:
  1. Simulates another TM M_L on the input to M'.
  2. Simulates M on w.
     - Note: neither M, M_L, nor w is input to M'.

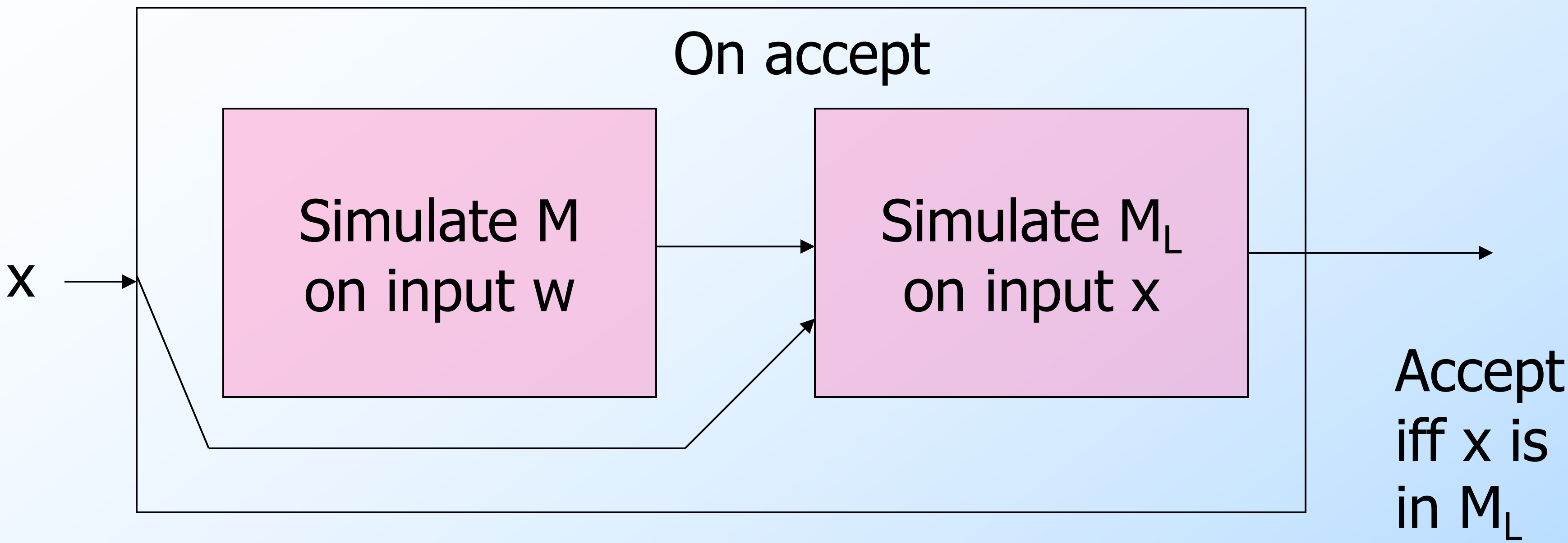
The Reduction – (2)

- Assume that $\emptyset$ does not have property P.
  - If it does, consider the complement of P, which would also be decidable by the lemma.
- Let L be any language with property P, and let M_L be a TM that accepts L.
- M' is constructed to work as follows (next slide).

Design of M'

1. On the second tape, write w and then simulate M on w.
2. If M accepts w, then simulate M_L on the input x to M', which appears initially on the first tape.
3. M' accepts its input x if and only if M_L accepts x.

Action of M' if M Accepts w



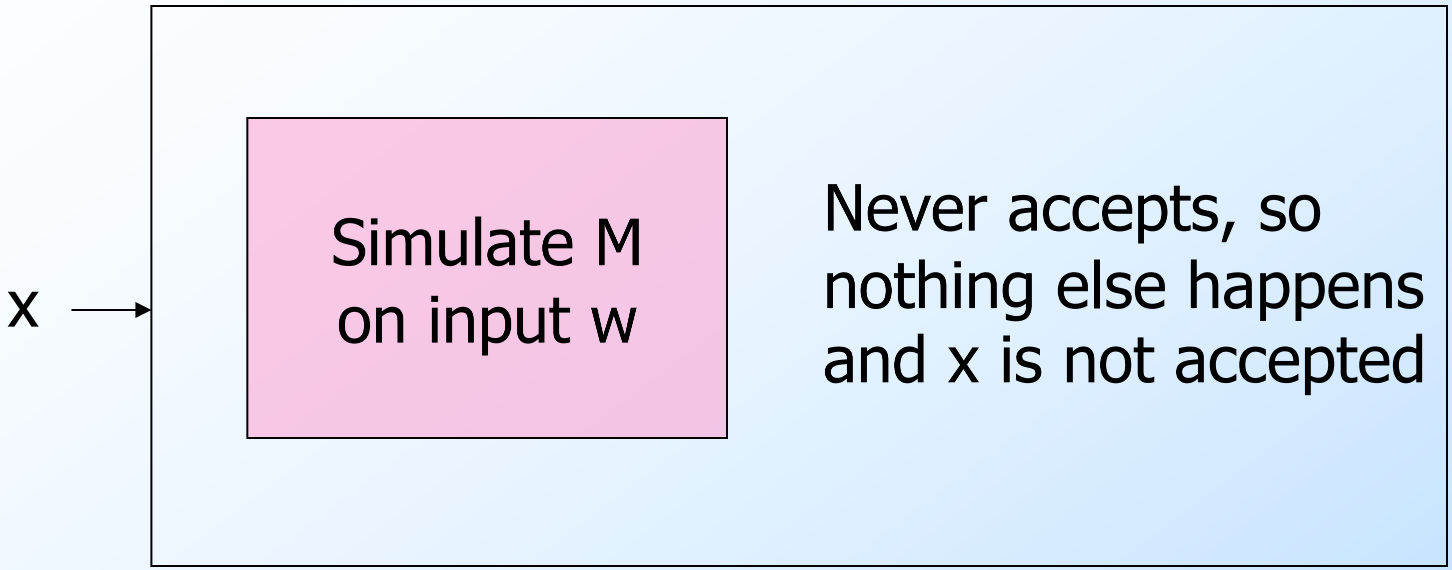
Design of M' – (2)

- Suppose M accepts w.
- Then M' simulates M_L and therefore accepts x if and only if x is in L.
- That is, L(M') = L, L(M') has property P, and M' is in L_P.

Design of M' – (3)

- Suppose M does not accept w.
- Then M' never starts the simulation of M_L, and never accepts its input x.
- Thus, L(M') = $\varnothing$, and L(M') does not have property P.
- That is, M' is not in L_P.

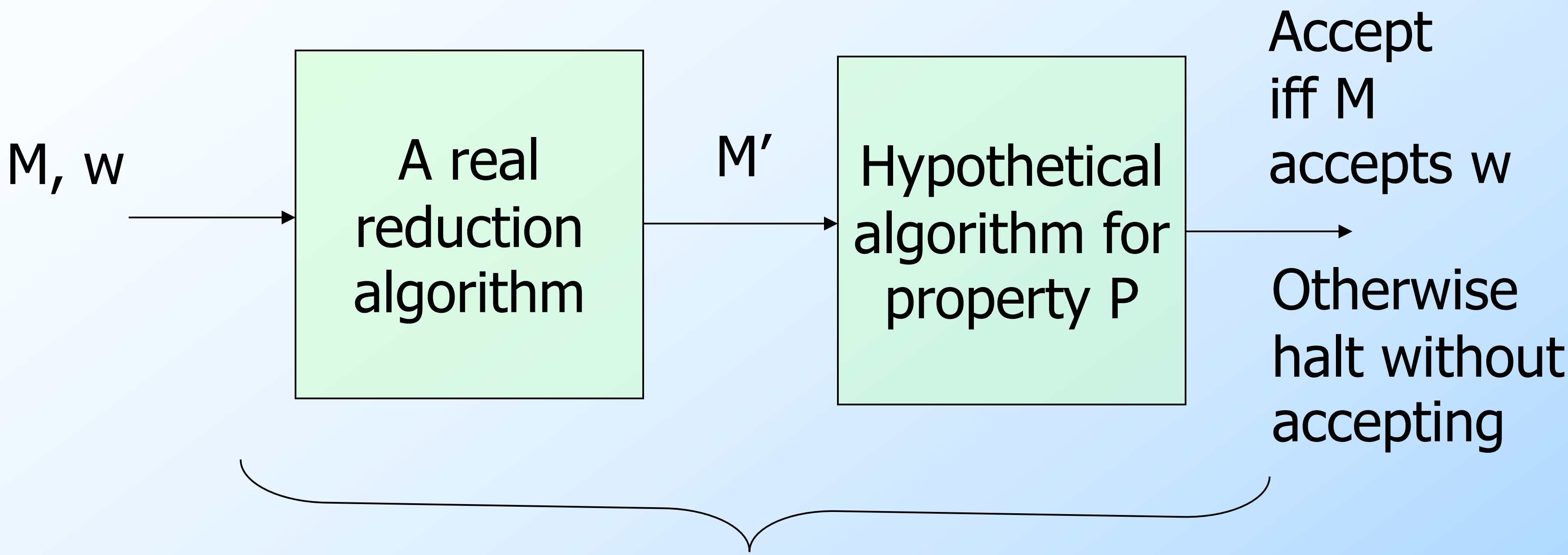
Action of M’ if M Does not Accept w



Design of M' – Conclusion

- Thus, the algorithm that converts M and w to M' is a reduction of L_u to L_P.
- Thus, L_P is undecidable.

Picture of the Reduction



This would be an algorithm
for L_u, which doesn’t exist

Applications of Rice's Theorem

- We now have any number of undecidable questions about TM's:
  - Is L(M) a regular language?
  - Is L(M) a CFL?
  - Does L(M) include any palindromes?
  - Is L(M) empty?
  - Does L(M) contain more than 1000 strings?
  - Etc., etc.