Network Models

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Some slides adapted from:
Networked Life (NETS) 112, Univ. of Penn., 2018, Prof. Michael Kearns
Efficient generation of large random networks, by Batagelj and Brandes.

Database of Real-World Graphs

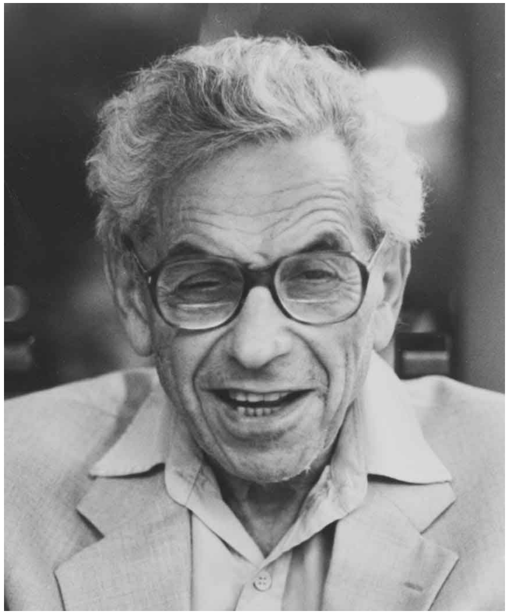
- **SNAP**: Stanford Network Analysis Project's Large Network Dataset Collection
- **http://snap.stanford.edu/data/index.html**
- Many real-world networks:
  - Social networks : online social networks, edges represent interactions between people
  - Communication networks : email communication networks with edges representing communication
  - Citation networks : nodes represent papers, edges represent citations
  - Collaboration networks : nodes represent scientists, edges represent collaborations (co-authoring a paper)
  - Web graphs : nodes represent webpages and edges are hyperlinks
  - Amazon networks : nodes represent products and edges link commonly co-purchased products
  - Internet networks : nodes represent computers and edges communication
  - … many more

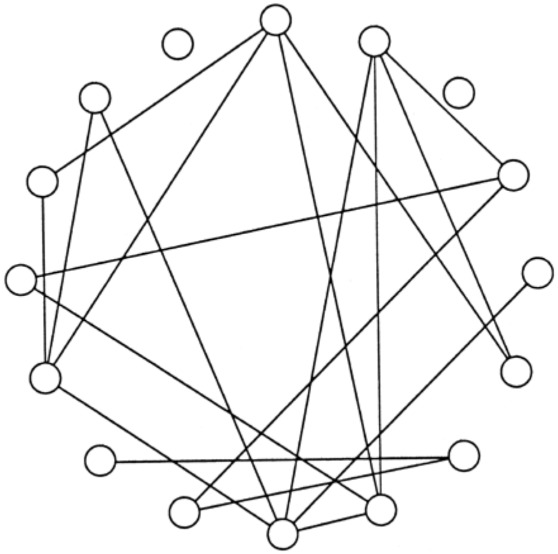
Network Models

- Recent studies of complex systems such as the Internet, biological networks, or social networks, have significantly increased the interest in modeling networks.
- Network models are desired that match real-world graph structures and properties, including:
  - Degree distributions
  - Small-world property
  - Clustering coefficients

Image from *https://matrix.berkeley.edu/research/social-networks-history*

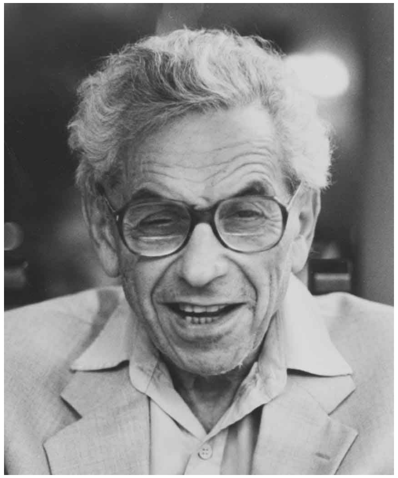
Network Models
I. The Erdös-Rényi (Random Graph) Model

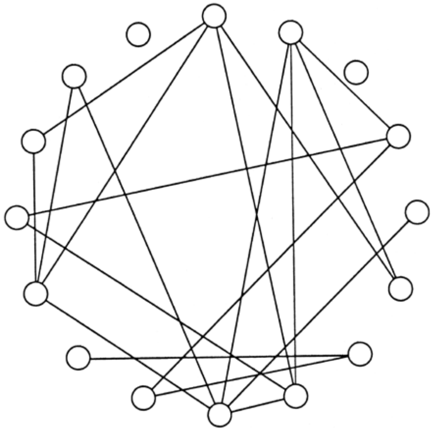




Random Graphs (Erdös/Rényi)

- **G(n,p)**:
  - n nodes
  - Every pair of nodes is connected independently with probability p
  - Average degree: d = (n-1)p ~ np

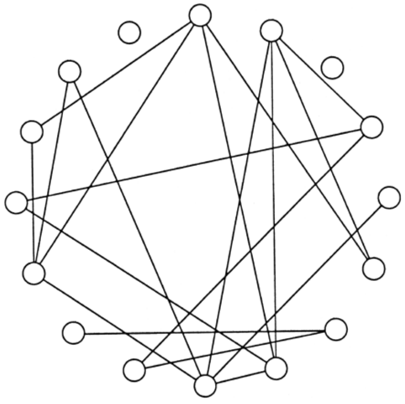




Erdös-Rényi G(n,p) Generation

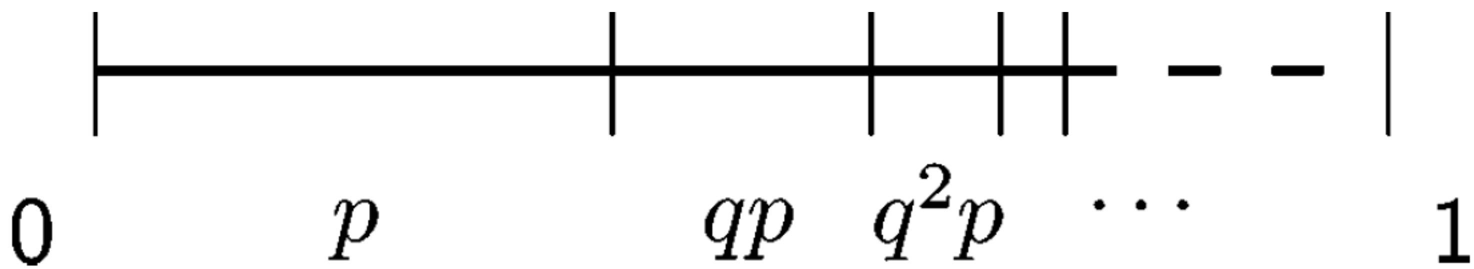
- Begin with n isolated vertices, no edges
- Consider (unordered) vertex pairs, {v,w}, according to some ordering.
- For each such pair, {v,w}:
  - Randomly generate a bit, b, that is 1 with probability p.
  - If b = 1, then add the edge (v,w) to the graph

- This algorithm runs in $O(n^2)$ time, however.



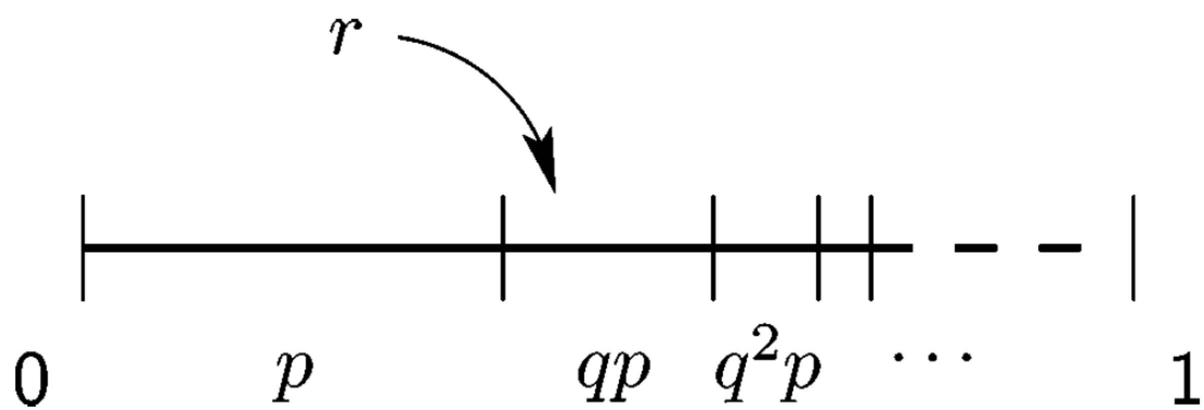
Faster Erdös-Rényi G(n,p) Generation

- The above algorithm for generating G(n,p) is slow if p is small, because most of the bits are 0.
- Probability of having k-1 0's then a 1 is $q^{k-1}p$, where $q = 1-p$.
- Waiting times are geometrically distributed.
- Divide the interval [0,1) according to the waiting times:



Faster Erdös-Rényi G(n,p) Generation

- Pick r uniformly at random in the interval [0,1)
- Divide the interval [0,1) according to the waiting times.
- The subinterval in which r falls will sample a waiting time:



Note that

$$r < 1 - q^k \Leftrightarrow k > \frac{\log(1-r)}{\log q},$$

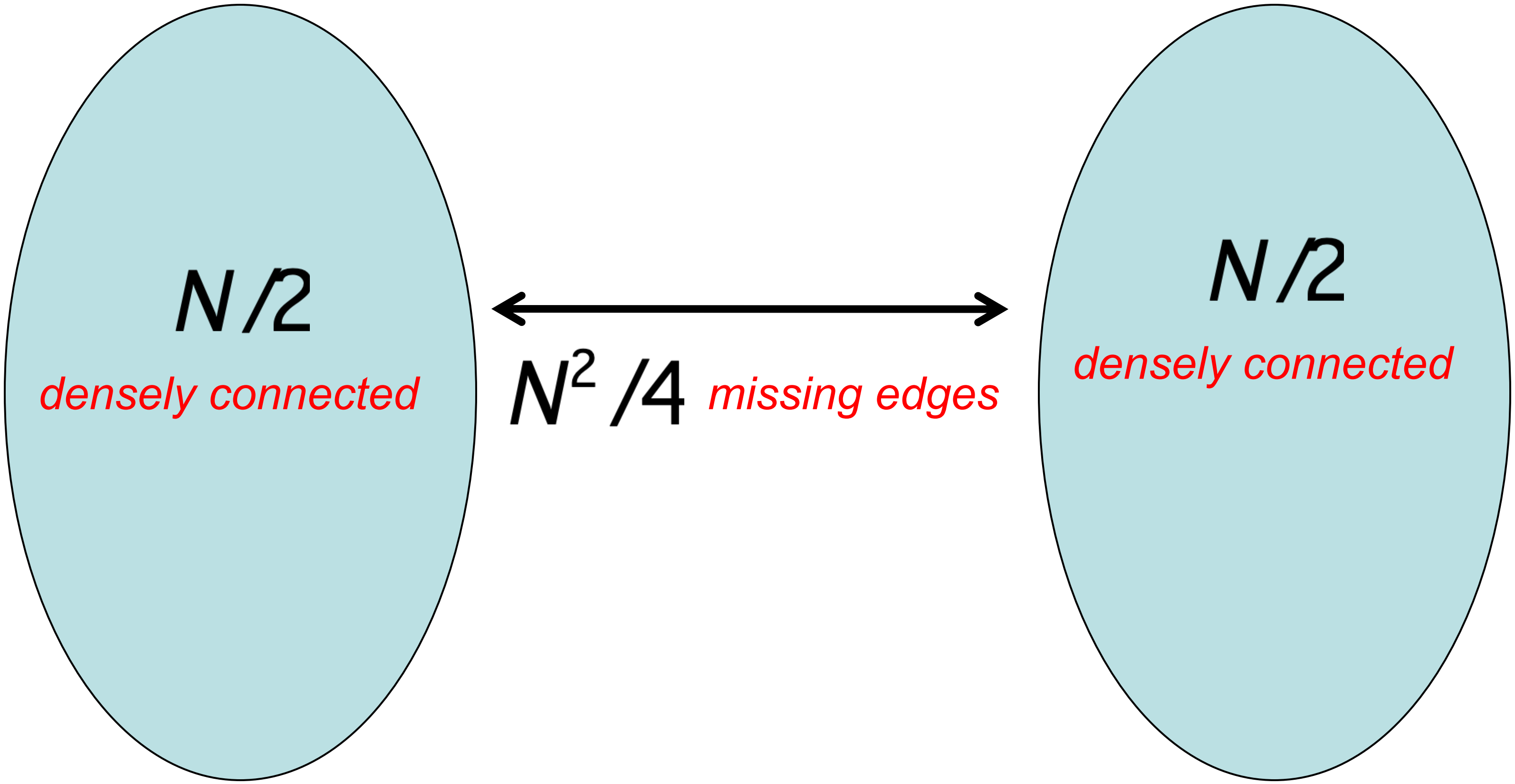
so that we choose $k = 1 + \lfloor \log(1-r)/\log q \rfloor$.

Faster Erdös-Rényi G(n,p) Generation

- The above algorithm for generating G(n,p) is slow if p is small, because most of the bits are 0.
- Probability of having k-1 0's then a 1 is $(1-p)^{k-1}p$
- Faster O(n+m)-time algorithm skips over runs of 0's:

ALG. 1: $\mathcal{G}(n,p)$
Input: number of vertices n, edge probability $0<p<1$
Output: $G=(\{0,\ldots,n-1\},E)\in\mathcal{G}(n,p)$
$E\leftarrow\emptyset$
$v\leftarrow 1;\ w\leftarrow -1$
while $v<n$ **do**
  draw $r\in[0,1)$ uniformly at random
  $w\leftarrow w+1+\lfloor\log(1-r)/\log(1-p)\rfloor$
  while $w\geqslant v$ **and** $v<n$ **do**
    $w\leftarrow w-v;\ v\leftarrow v+1$
  if $v<n$ **then** $E\leftarrow E\cup\{v,w\}$

There Can't Be Two Large Components?



Threshold Phenomena in Erdös-Renyi

- Theorem: In Erdös-Renyi, as n becomes large:
  - If $p < 1/n$, probability of a giant component (e.g. 50% of vertices) goes to 0
  - If $p > 1/n$, probability of a giant component goes to 1, and all other components will have size at most log(n)
- Note: at edge density p, expected/average degree is $p(N-1) \sim pn$
- So at $p \sim 1/n$, average degree is ~ 1: incredibly sparse
- So model "explains" giant components in real networks
- General "tipping point" at edge density q (may depend on n):
  - If $p < q$, probability of property goes to 0 as n becomes large
  - If $p > q$, probability of property goes to 1 as n becomes large
- For example, could examine property "diameter 6 or less"

Threshold Phenomena in Erdös-Renyi

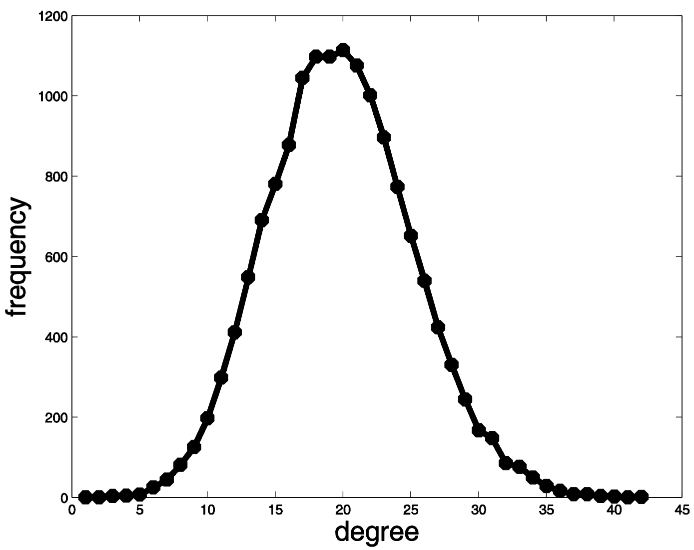
- Theorem: In Erdös-Renyi, as N becomes large:
  - The diameter is O(log (N) / log (Np).
  - Threshold at

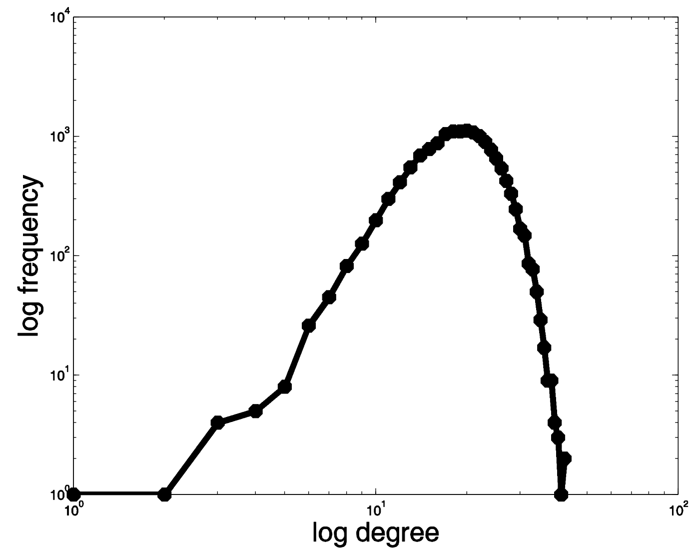
$$p \sim \log(N)/N^{5/6}$$

  - for diameter 6.
  - Note: degrees growing (slightly) with N
  - If N = 300M (U.S. population) then average degree pN ~ 500
  - If N = 7BN (world population) then average degree pN ~ 1000
  - Not unreasonable figures…
- At p not too far from 1/N, get strong connectivity
- Very efficient use of edges

What Doesn't the Model Explain?

- Erdös-Renyi explains giant component and small diameter
- But:
  - degree distribution not heavy-tailed; exponential decay from mean (Poisson)
  - clustering coefficient is *exactly* p





- To model these real-world phenomena, we'll need **richer** models with greater realism…

Rich-Get-Richer Processes

- Processes in which the more someone has of something, the more likely they are to get more of it
- Examples:
  - the more friends you have, the easier it is to make more
  - the more business a firm has, the easier it is to win more
  - the more people there are at a nightclub, the more who want to go
- Such processes will amplify inequality
- One simple and general model: if you have amount x of something, the probability you get more is proportional to x
  - so if you have twice as much as me, you're twice as likely to get more
- Generally leads to heavy-tailed distributions (power laws)

Preferential Attachment

- Start with two vertices connected by an edge
- At each step, add one *new* vertex v with one edge back to *previous* vertices
- Probability a previously added vertex u receives the new edge from v is *proportional to the (current) degree of u*
  - more precisely, probability u gets the edge is

    (current degree of u)/(sum of all current degrees)
- Vertices with high degree are likely to get *even more* links!
  - just like Instagram, Twitter, …
- *Generates a power law distribution of degrees*
- Variation: each new vertex initially gets d edges

Barabasi-Albert (BA) model

- The BA model for preferential attachment
  - input: some initial subgraph G_0, and d the number of edges per new node
  - the process:
    - nodes arrive one at the time
    - each node connects to d other nodes selecting them with probability proportional to their degree
    - if $[d_1,\ldots,d_t]$ is the degree sequence at time t, the node $t+1$ links to node i with probability equal to

$$\frac{d_i}{\sum_i d_i}$$

- Guarantees a degeneracy of d. Why?
- Brute-force algorithm runs in $O(n^2)$ time. (Bad.)

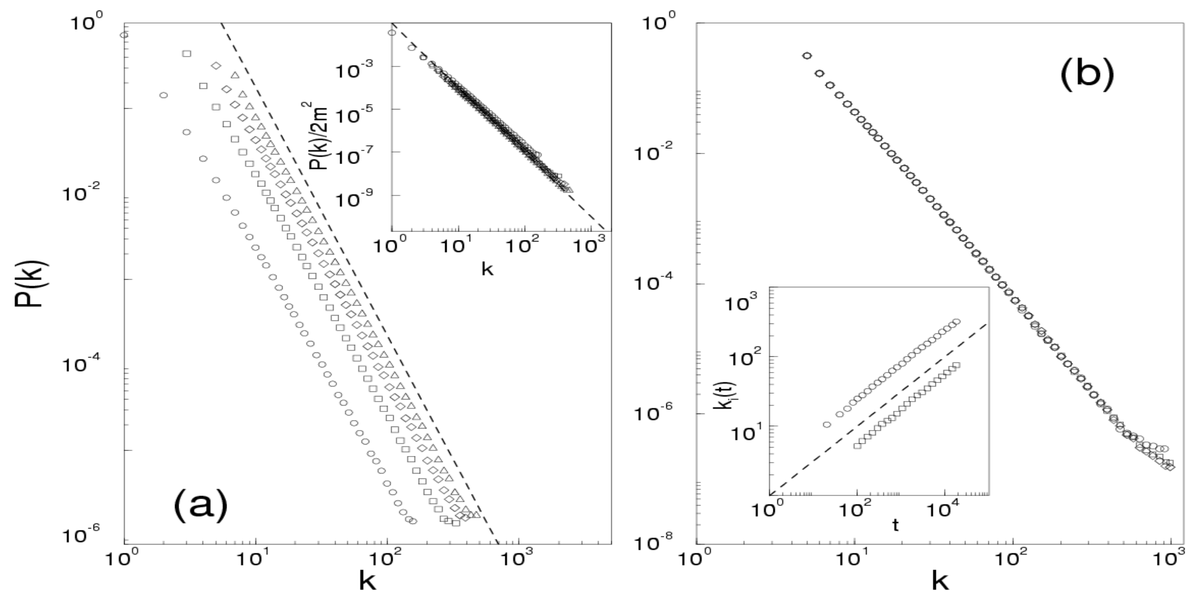
Faster Barabasi-Albert (BA) Algorithm

- Let d be the parameter for the BA algorithm

ALG. 5: preferential attachment
Input: number of vertices n
minimum degree $d \geqslant 1$
Output: scale-free multigraph
$G = (\{0, \ldots, n-1\}, E)$
M: array of length $2nd$ // M is an array of edges chosen so far.
for $v = 0, \ldots, n-1$ **do**
  for $i = 0, \ldots, d-1$ **do**
    $M[2(vd+i)] \leftarrow v$ // Each vertex v appears d_v times in M.
    draw $r \in \{0, \ldots, 2(vd+i)\}$ uniformly at random
    $M[2(vd+i)+1] \leftarrow M[r]$
$E \leftarrow \emptyset$
for $i = 0, \ldots, nd-1$ **do**
  $E \leftarrow E \cup \{M[2i], M[2i+1]\}$

Barabasi-Albert (BA) algorithm

- Faster algorithm runs in O(nd) = O(n+m) time.
- The BA model should result in power-law degree distribution with exponent c = -3



c = -3. different d’s. P(k) changes.

c does not