Random Permutations

Trees with snow on branches, "Half Dome, Apple Orchard, Yosemite," 1933. Ansel Adams. U.S. government image. U.S. National Archives and Records Administration.

Generating Random Permutations

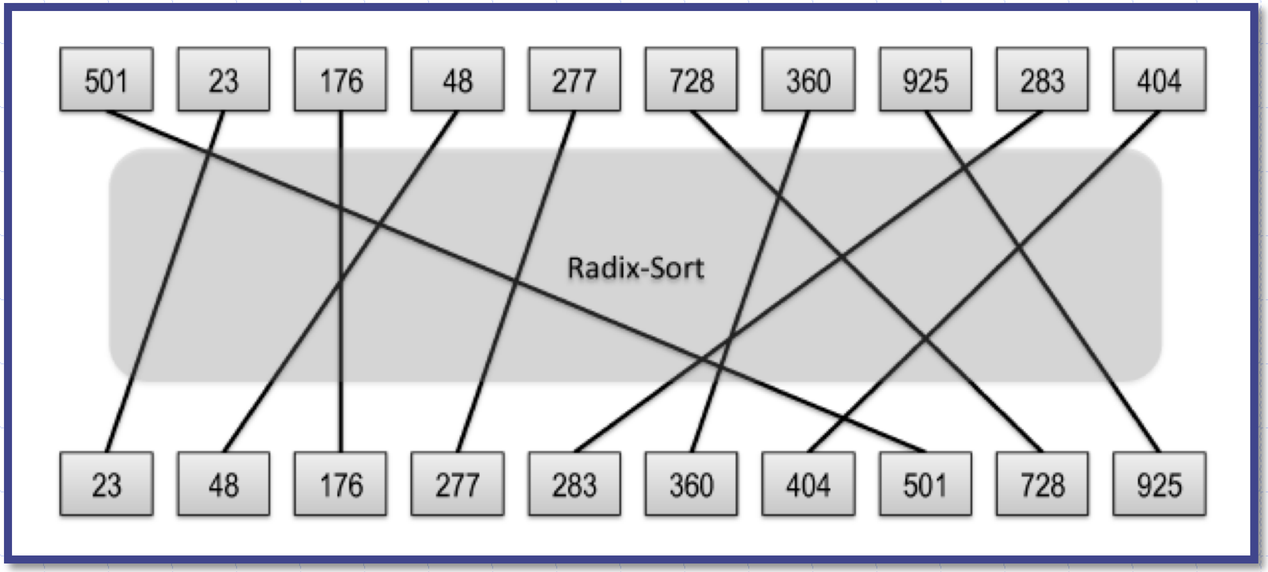
- The input to the random permutation problem is a list, $X = (x_1, x_2, \ldots, x_n)$, of n elements, which could stand for playing cards or any other objects we want to randomly permute.
- The output is a reordering of the elements of X, done in a way so that all permutations of X are equally likely.
- We can use a function, **random**(k), which returns an integer in the range $[0, k - 1]$ chosen uniformly and independently at random.

Applications: Simple Algorithms and Card Games

- A **randomized algorithm** is an algorithm whose behavior depends, in part, on the outcomes of random choices or the values of random bits.
- The main advantage of using randomization in algorithm design is that the results are often simple and efficient.
- In addition, there are some problems that need randomization for them to work effectively.
- For instance, consider the problem common in computer games involving playing cards—that of randomly shuffling a deck of cards so that all possible orderings are equally likely.

Algorithm 1: Random Sort

- This algorithm simply chooses a random number for each element in X and sorts the elements using these values as keys.



Analysis of Random-Sort

- To see that every permutation is equally likely to be output by the random-sort method, note that each element, x_i, in X has an equal probability, 1/n, of having its random r_i value be the smallest.
- Thus, each element in X has equal probability of 1/n of being the first element in the permutation.
- Applying this reasoning recursively, implies that the permutation that is output has the following probability of being chosen:

$$\left(\frac{1}{n}\right)\cdot\left(\frac{1}{n-1}\right)\cdots\left(\frac{1}{2}\right)\cdot\left(\frac{1}{1}\right) = \frac{1}{n!}$$

- That is, each permutation is equally likely to be output.
- There is a small probability that this algorithm will fail, however, if the random values are not unique.

Fisher-Yates Shuffling

- There is a different algorithm, known as the Fisher-Yates algorithm, which always succeeds.

Algorithm FisherYates(X):

Input: An array, X, of n elements, indexed from position 0 to $n-1$

Output: A permutation of X so that all permutations are equally likely

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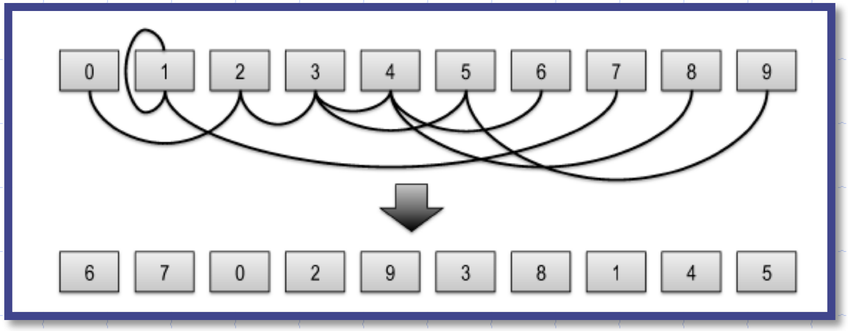
for k = n − 1 downto 1 do
    Let j ← random(k + 1)      // j is a random integer in [0, k]
    Swap X[k] and X[j]         // This may "swap" X[k] with itself, if j = k
return X
```



Analysis of Fisher-Yates

- This algorithm considers the items in the array one at time from the end and swaps each element with an element in the array from that point to the beginning.
- Notice that each element has an equal probability, of 1/n, of being chosen as the last element in the array X (including the element that starts out in that position).
- Applying this analysis recursively, we see that the output permutation has probability

$$\left(\frac{1}{n}\right) \cdot \left(\frac{1}{n-1}\right) \cdots \left(\frac{1}{2}\right) \cdot \left(\frac{1}{1}\right) = \frac{1}{n!}$$

- That is, each permutation is equally likely.