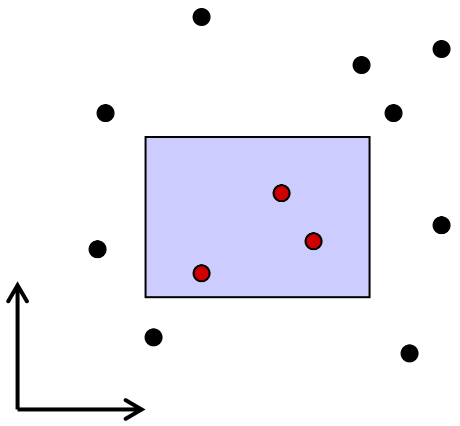
Computational Geometry



Orthogonal Range Searching

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with slides from Carola Wenk

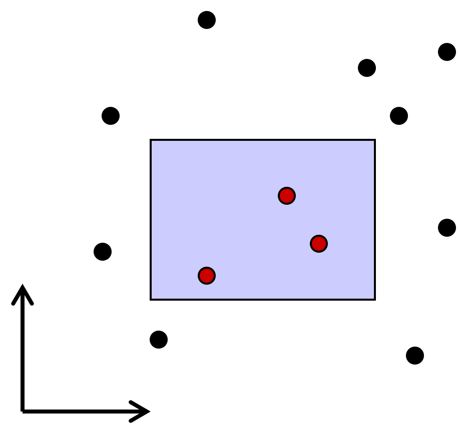
Orthogonal range searching

Input: n points in d dimensions

- E.g., representing a database of n records each with d numeric fields

Query: Axis-aligned ***box*** (in 2D, a rectangle)

- Report on the points inside the box:
  - Are there any points?
  - How many are there?
  - List the points.



Orthogonal range searching

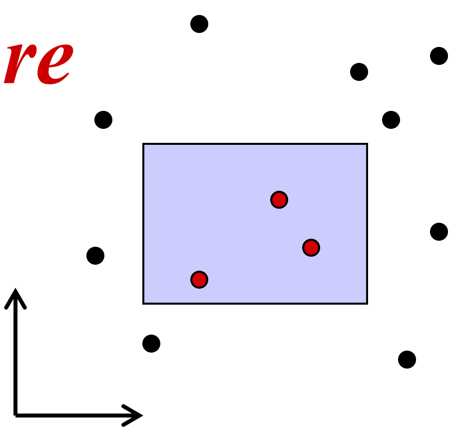
Input: n points in d dimensions

Query: Axis-aligned ***box*** (in 2D, a rectangle)

- Report on the points inside the box

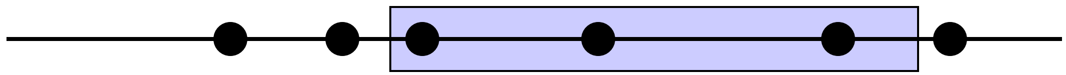
Goal: Preprocess points into a data structure
to support fast queries

- Primary goal: ***Static data structure***
- In 1D, we will also obtain a dynamic data structure supporting insert and delete



1D range searching

In 1D, the query is an interval:



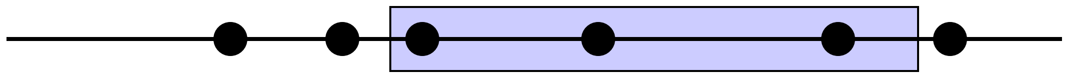
First solution:

- Sort the points and store them in an array
  - Solve query by binary search on endpoints.
  - Obtain a static structure that can list k answers in a query in $O(k + \log n)$ time.

Goal: Obtain a dynamic structure that can list k answers in a query in $O(k + \log n)$ time.

1D range searching

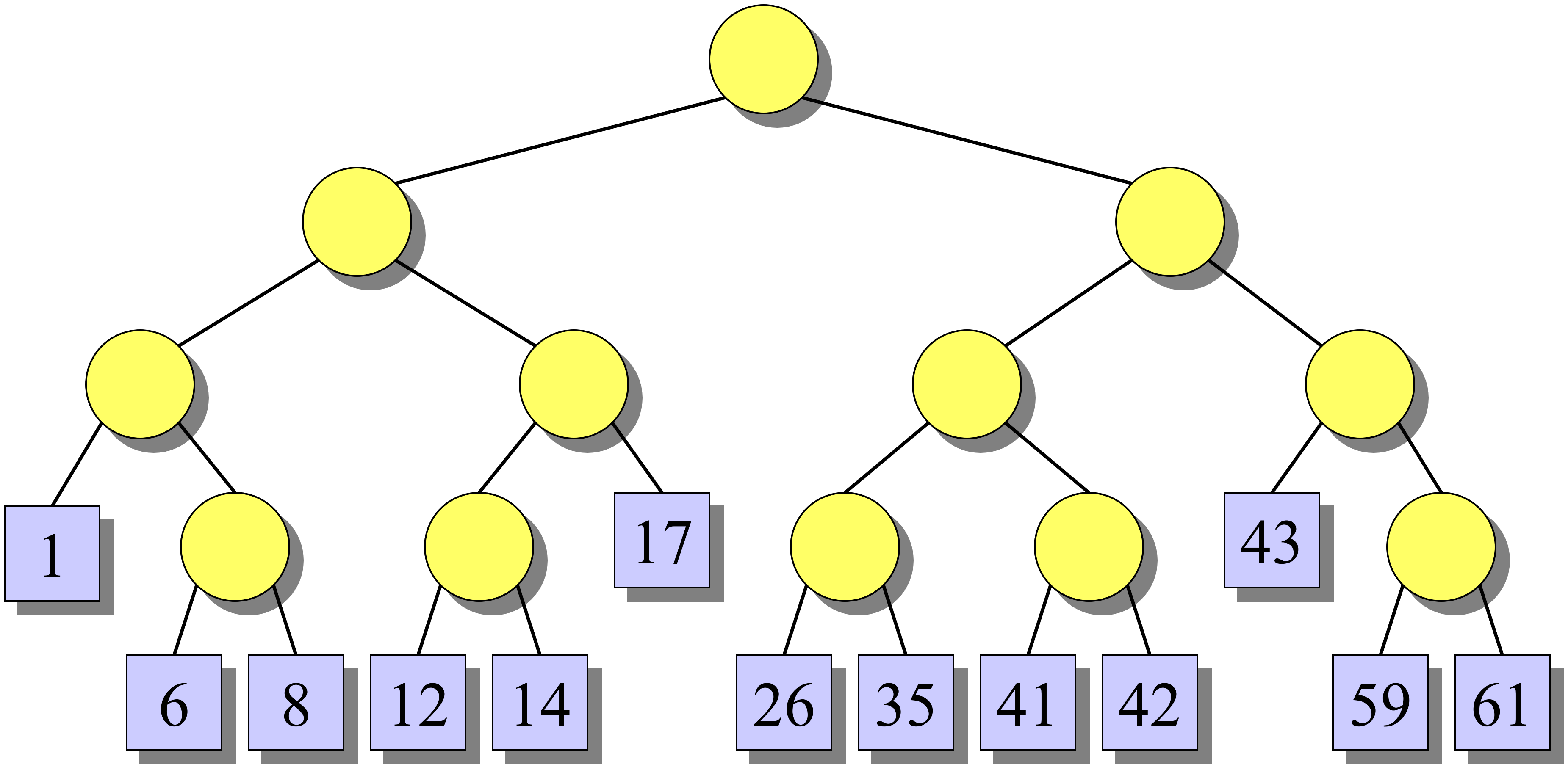
In 1D, the query is an interval:



New solution that extends to higher dimensions:

- Balanced binary search tree
  - New organization principle:
    Store points in the *leaves* of the tree.
  - Internal nodes store copies of the leaves to satisfy binary search property:
    - Node x stores in $key[x]$ the maximum key of any leaf in the left subtree of x.

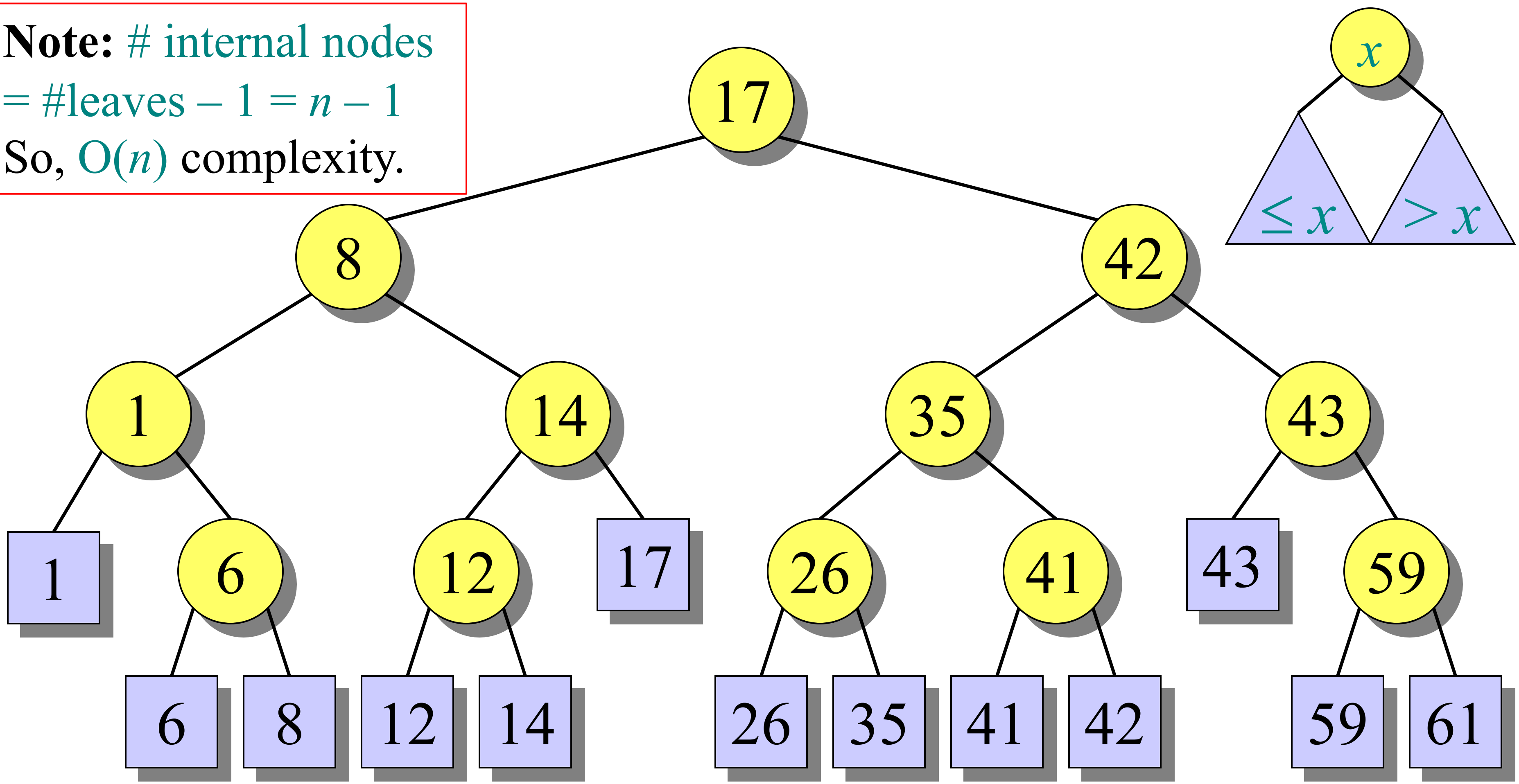
Example of a 1D range tree



key[*x*] is the maximum key of any leaf in the left subtree of *x*.

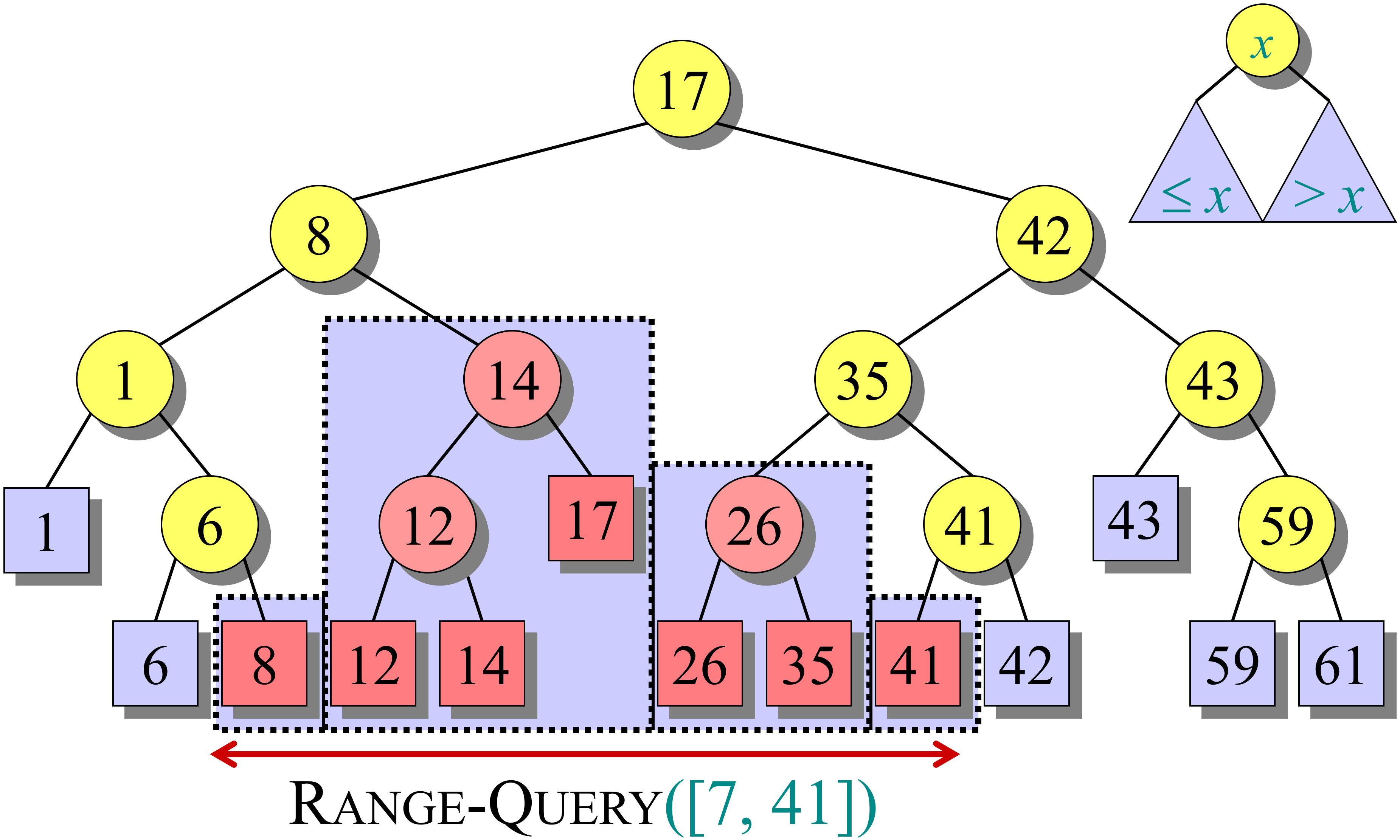
Example of a 1D range tree

Note: # internal nodes = #leaves – 1 = n – 1
So, O(n) complexity.

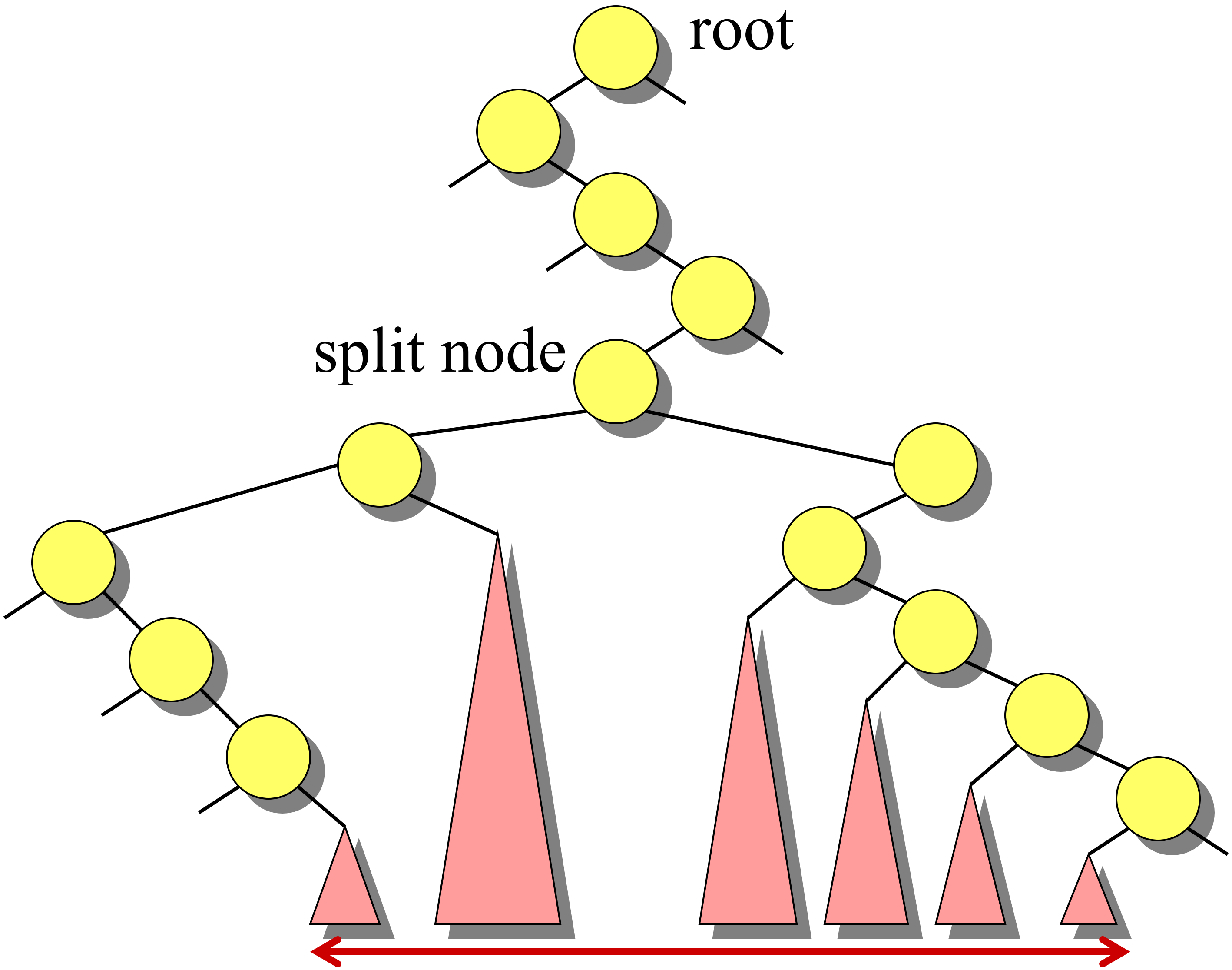


key[x] is the maximum key of any leaf in the left subtree of x.

Example of a 1D range query



General 1D range query



Pseudocode, part 1: Find the split node

1D-RANGE-QUERY$(T, [x_1, x_2])$

$w \leftarrow T$.root

while w is not a leaf and $(x_2 \leq w.key$ or $w.key < x_1)$

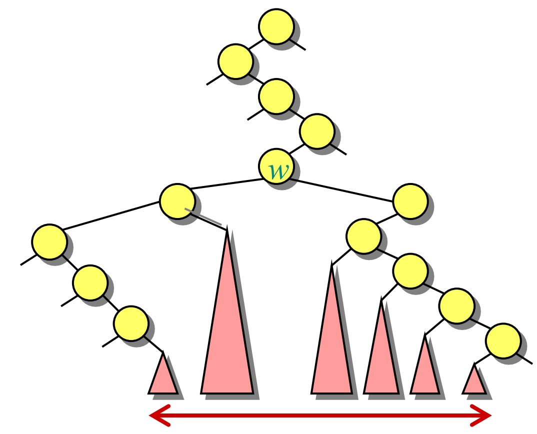
do if $x_2 \leq w.key$

then $w \leftarrow w.left$

else $w \leftarrow w.right$

// w is now the split node

[*traverse left and right from* w *and report relevant subtrees*]



Pseudocode, part 2: Traverse left and right from split node

1D-RANGE-QUERY$(T, [x_1, x_2])$

[find the split node]

// w is now the split node

if w is a leaf

then output the leaf w if $x_1 \leq w.key \leq x_2$

else $v \leftarrow w.left$ // Left traversal

while v is not a leaf

do if $x_1 \leq w.key$

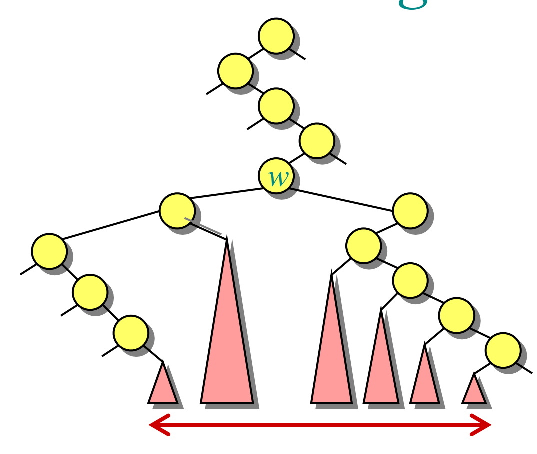
then output all leaves in the subtree rooted at $v.right$

$v \leftarrow v.left$

else $v \leftarrow v.right$

output the leaf v if $x_1 \leq v.key \leq x_2$

[symmetrically for right traversal]

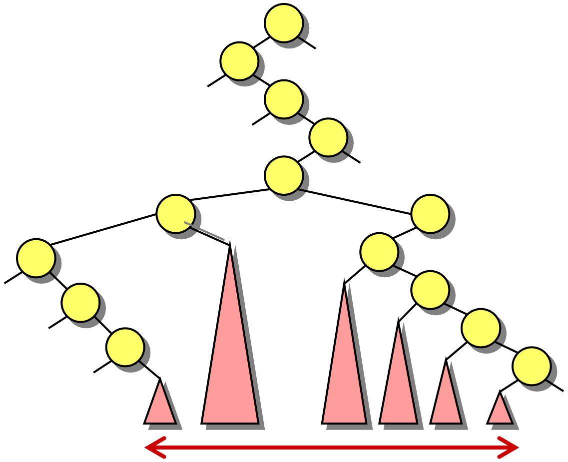


Analysis of 1D-Range-Query

Query time: Answer to range query represented by $O(\log n)$ subtrees found in $O(\log n)$ time.

Thus:

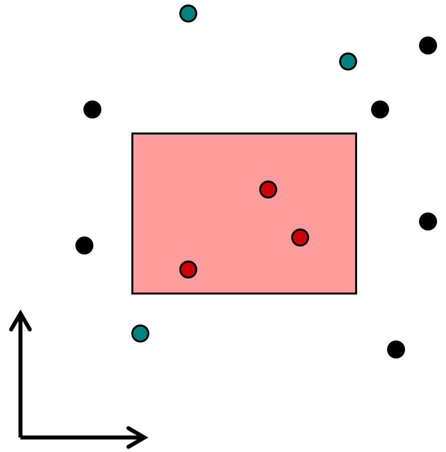
- Can test for points in interval in $O(\log n)$ time.
- Can report all k points in interval in $O(k + \log n)$ time.
- Can count points in interval in $O(\log n)$ time



Space: $O(n)$

Preprocessing time: $O(n \log n)$

2D range trees

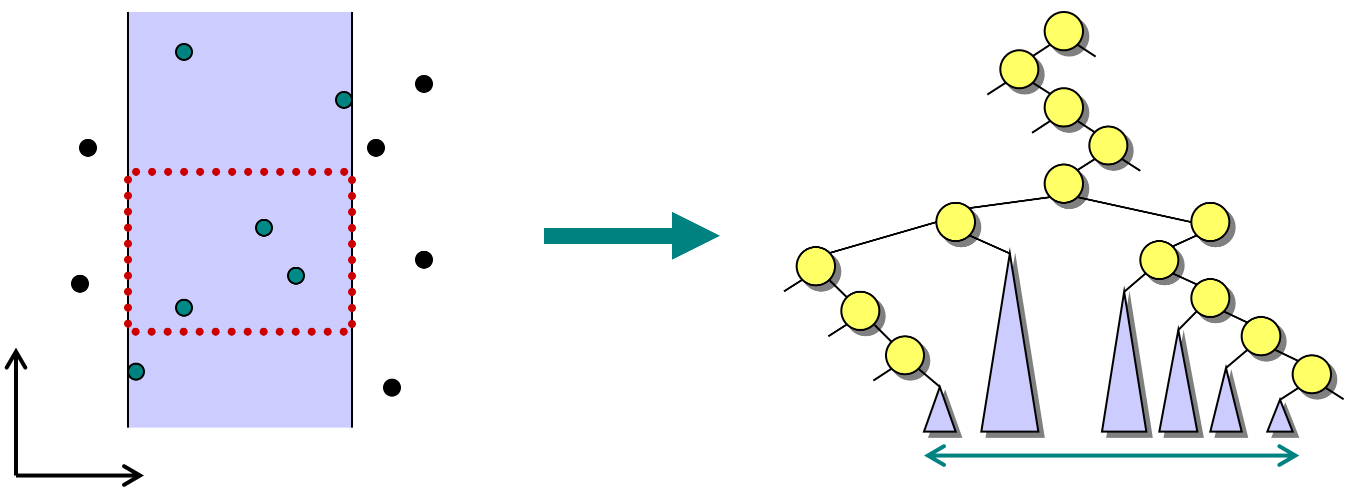


2D range trees

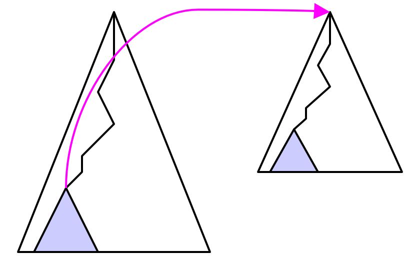
Store a ***primary*** 1D range tree for all the points based on *x*-coordinate.

Thus in O(log *n*) time we can find O(log *n*) subtrees representing the points with proper *x*-coordinate.

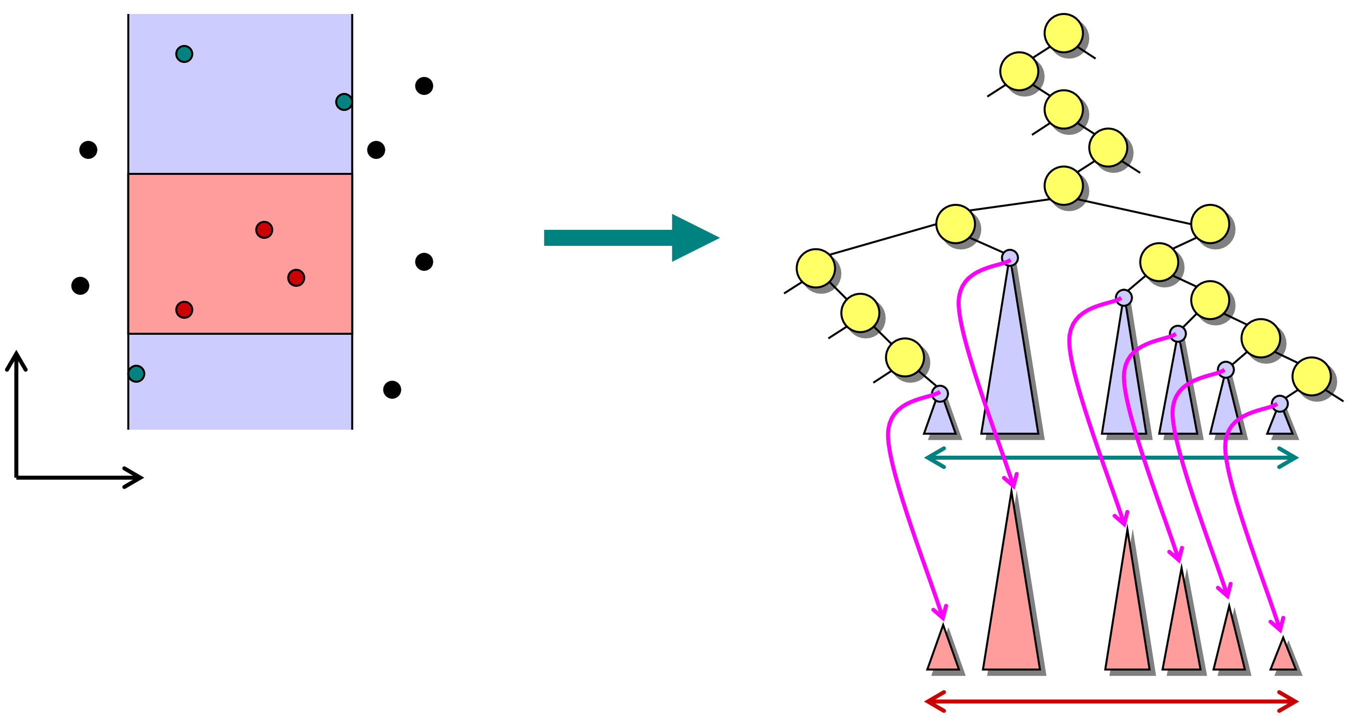
How to restrict to points with proper *y*-coordinate?



2D range trees

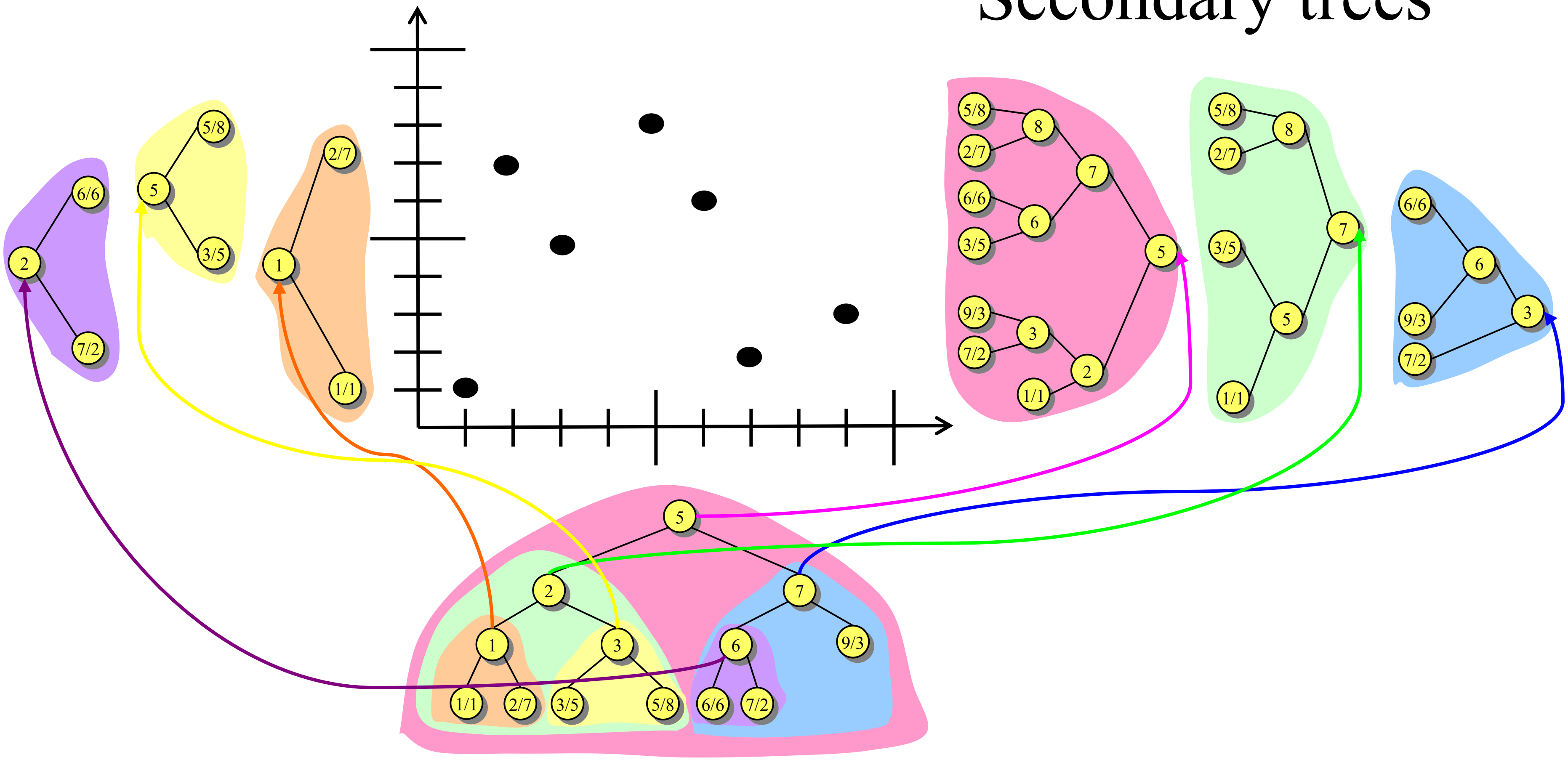


Idea: In primary 1D range tree of *x*-coordinate, **every** node stores a ***secondary*** 1D range tree based on *y*-coordinate for all points in the subtree of the node. Recursively search within each.



2D range tree example

Secondary trees



Primary tree

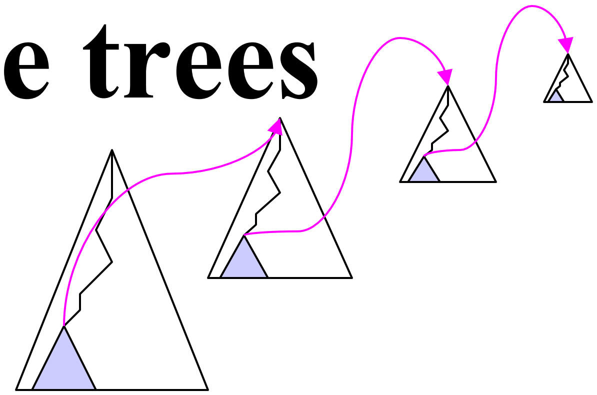
Analysis of 2D range trees

Query time: In $O(\log^2 n) = O((\log n)^2)$ time, we can represent answer to range query by $O(\log^2 n)$ subtrees. Total cost for reporting k points: $O(k + (\log n)^2)$.

Space: The secondary trees at each level of the primary tree together store a copy of the points. Also, each point is present in each secondary tree along the path from the leaf to the root. Either way, we obtain that the space is $O(n \log n)$.

Preprocessing time: $O(n \log n)$

d-dimensional range trees



Each node of the secondary y-structure stores a tertiary z-structure representing the points in the subtree rooted at the node, etc.

Save one log factor using fractional cascading

Query time: $O(k + \log^d n)$ to report k points.
Space: $O(n \log^{d-1} n)$
Preprocessing time: $O(n \log^{d-1} n)$

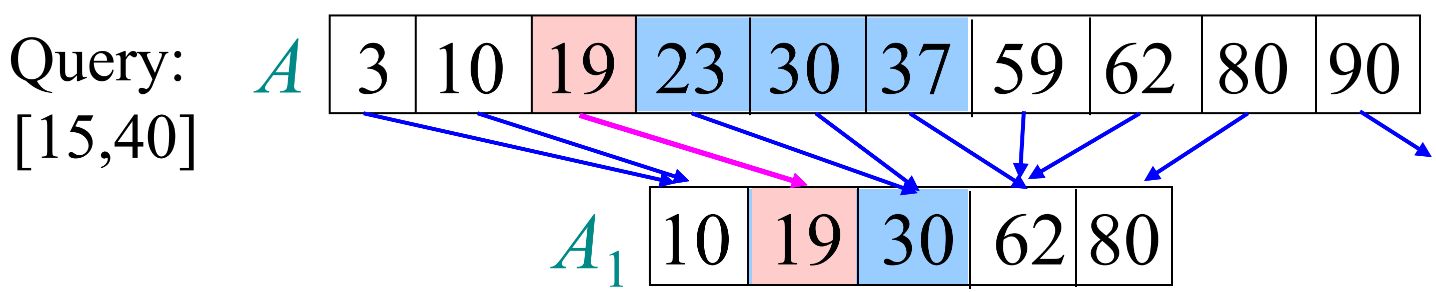
Search in Subsets

Given: Two sorted arrays A_1 and A, with $A_1 \subseteq A$
A query interval $[l,r]$

Task: Report all elements e in A_1 and A with $l \leq e \leq r$

Idea: Add pointers from A to A_1:
$\rightarrow$ For each $a \in A$ add a pointer to the smallest element $b \in A_1$ with $b \geq a$

Query: Find $l \in A$, follow pointer to A_1. Both in A and A_1 sequentially output all elements in $[l,r]$.

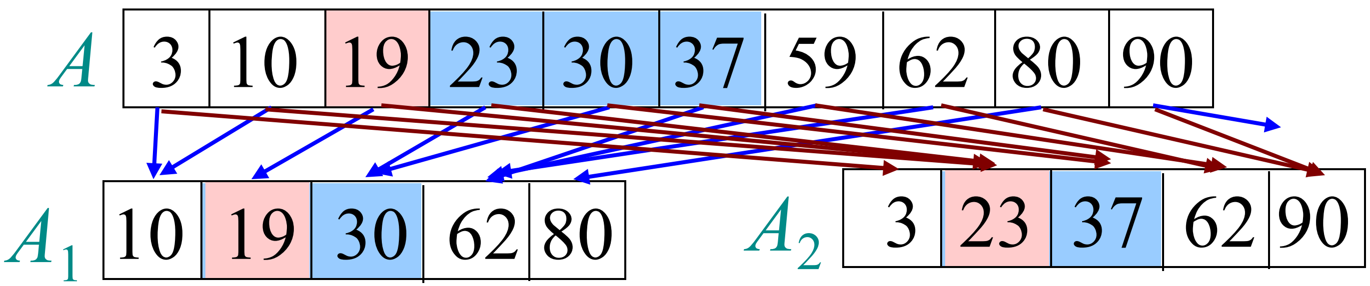


Runtime: $O((\log n + k) + (1 + k)) = O(\log n + k)$

Search in Subsets (cont.)

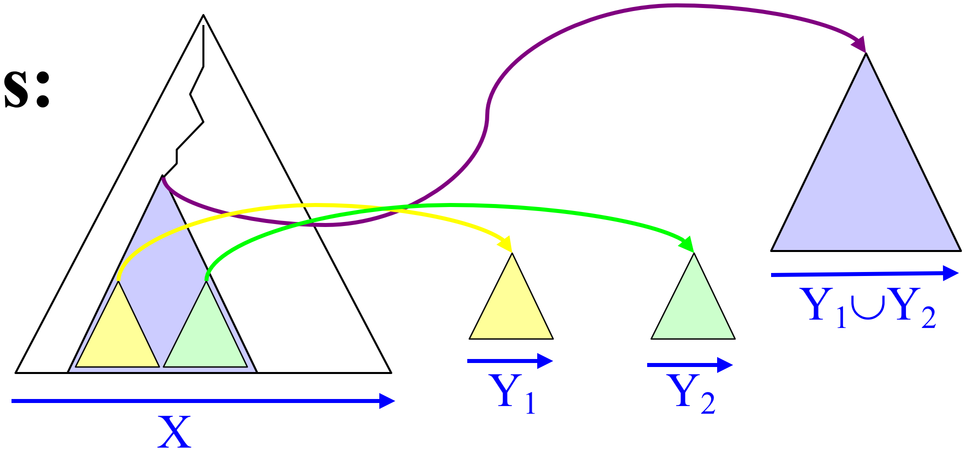
Given: Three sorted arrays A_1, A_2, and A, with $A_1 \subseteq A$ and $A_2 \subseteq A$

Query: [15,40]



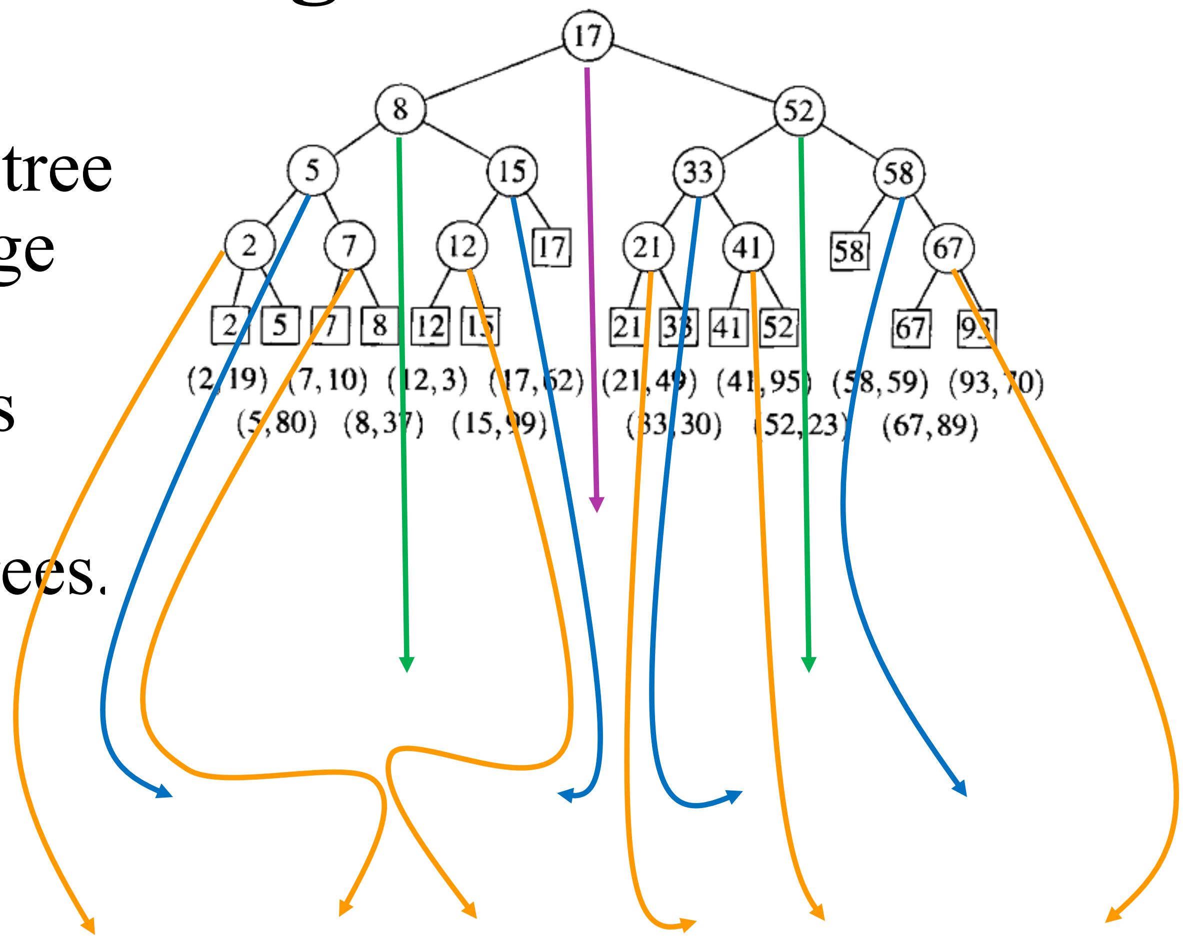
Runtime: $O((\log n + k) + (1+k) + (1+k)) = O(\log n + k))$

Range trees:



Fractional Cascading: Layered Range Tree

Replace 2D range tree with a layered range tree, using sorted arrays and pointers instead of the secondary range trees.



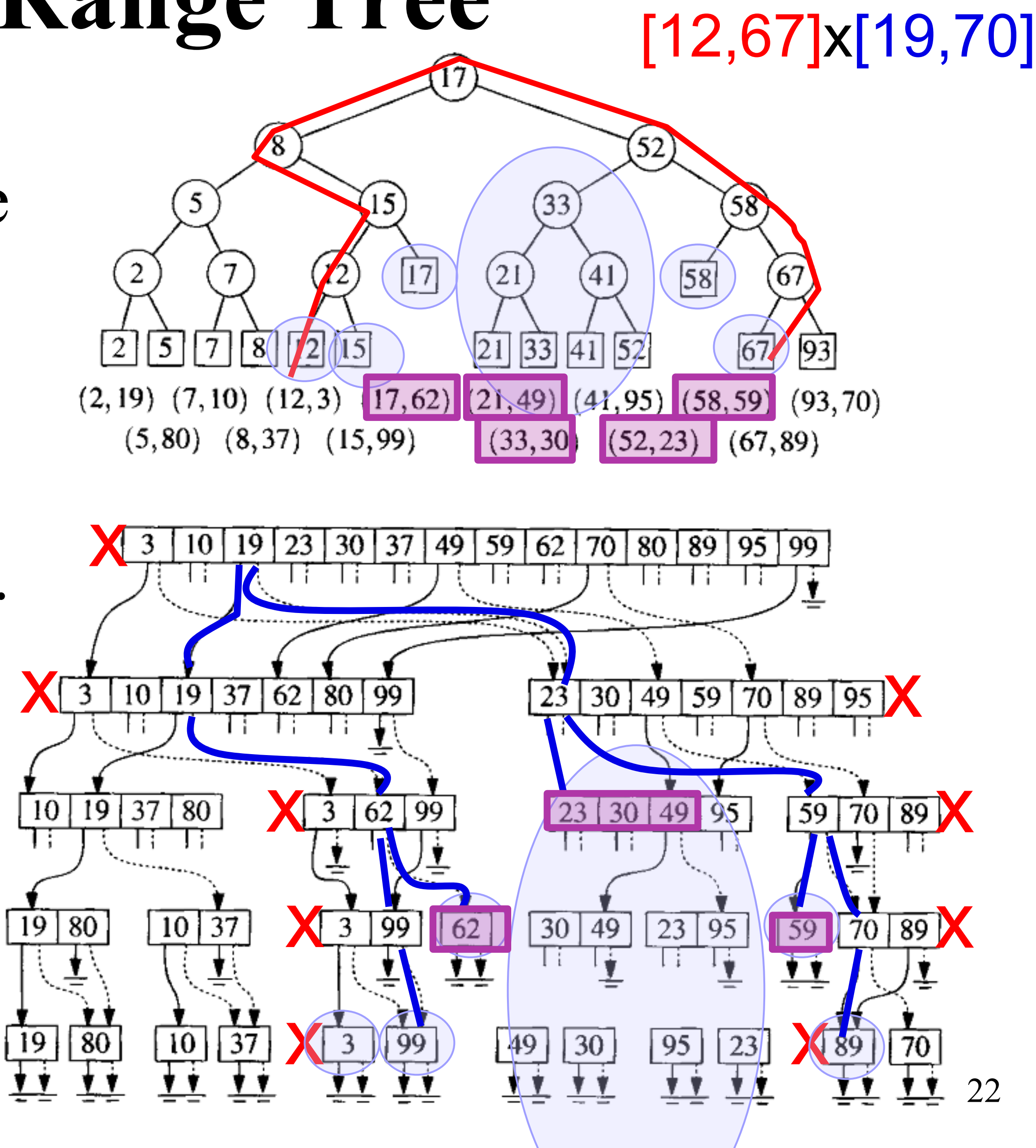
Preprocessing:

$O(n \log n)$

Query:

$O(\log n + k)$

Fractional Cascading: Layered Range Tree



Replace 2D range tree with a layered range tree, using sorted arrays and pointers instead of the secondary range trees.

Preprocessing:

$O(n \log n)$

Query:

$O(\log n + k)$

d-dimensional range trees

Query time: $O(k + \log^{d-1} n)$ to report k points, uses fractional cascading in the last dimension
Space: $O(n \log^{d-1} n)$
Preprocessing time: $O(n \log^{d-1} n)$

Best data structure to date:
Query time: $O(k + \log^{d-1} n)$ to report k points.
Space: $O(n (\log n / \log \log n)^{d-1})$
Preprocessing time: $O(n \log^{d-1} n)$