Presentation for use with the textbook, Algorithm Design and Applications, by M. T. Goodrich and R. Tamassia, Wiley, 2015

Union-Find Structures

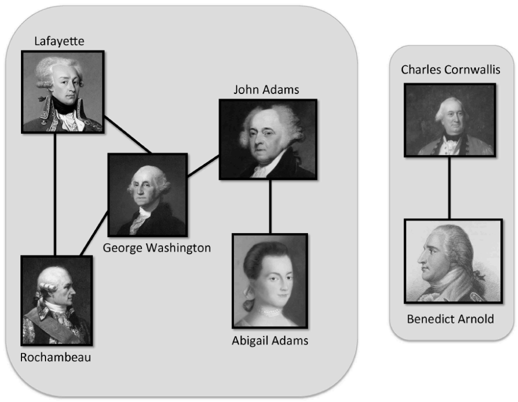
Merging galaxies, NGC 2207 and IC 2163. Combined image from NASA's Spitzer Space Telescope and Hubble Space Telescope. 2006. U.S. government image. NASA/JPL-Caltech/STScI/Vassar.

Application: Connected Components in a Social Network

- Social networking research studies how relationships between various people can influence behavior.
- Given a set, S, of n people, we can define a social network for S by creating a set, E, of edges or ties between pairs of people that have a certain kind of relationship. For example, in a friendship network, like Facebook, ties would be defined by pairs of friends.
- A **connected component** in a friendship network is a subset, T, of people from S that satisfies the following:
  - Every person in T is related through friendship, that is, for any x and y in T, either x and y are friends or there is a chain of friendship, such as through a friend of a friend of a friend, that connects x and y.
  - No one in T is friends with anyone outside of T.

Example

- 2 Connected components in a friendship network of some of the key figures in the American Revolutionary War.



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Union-Find Operations

- A **partition** or **union-find** structure is a data structure supporting a collection of disjoint sets subject to the following operations:
- makeSet(e): Create a singleton set containing the element e and return the position storing e in this set
- union(A,B): Return the set A U B, naming the result "A" or "B"
- find(e): Return the set containing the element e

Connected Components Algorithm

- The output from this algorithm is an identification, for each person x in S, of the connected component to which x belongs.

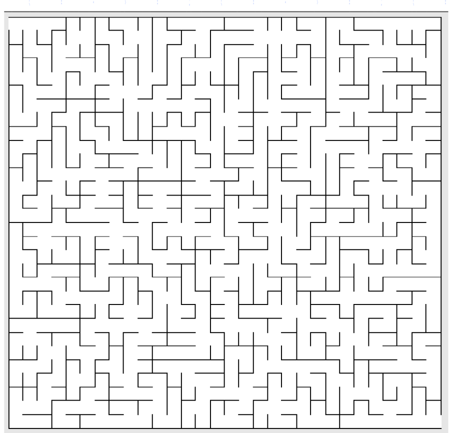
Algorithm UFConnectedComponents(S, E):
 Input: A set, S, of n people and a set, E, of m pairs of people from S defining pairwise relationships
 Output: An identification, for each x in S, of the connected component containing x

for each x in S **do**
  makeSet(x)
for each (x, y) in E **do**
  if find$(x) \neq$ find(y) **then**
    union(find(x), find(y))
for each x in S **do**
  Output "Person x belongs to connected component" find(x)

- The running time of this algorithm is O(t(n,n+m)), where t(j,k) is the time for k union-find operations starting from j singleton sets.

Another Application: Maze Construction and Percolation

- Problem: Construct a good maze.



A Maze Generator

Algorithm MazeGenerator(G, E):
  Input: A grid, G, consisting of n cells and a set, E, of m "walls," each of which divides two cells, x and y, such that the walls in E initially separate and isolate all the cells in G
  Output: A subset, R of E, such that removing the edges in R from E creates a maze defined on G by the remaining walls

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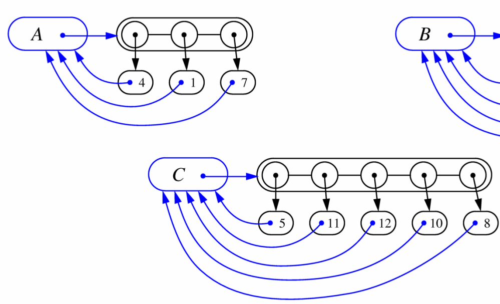
while R has fewer than n − 1 edges do
    Choose an edge, (x, y), in E uniformly at random from among those previously unchosen
    if find(x) ≠ find(y) then
        union(find(x), find(y))
        Add the edge (x, y) to R
return R
```

- This is actually is related to the science of **percolation theory**, which is the study of how liquids permeate porous materials.
  - For instance, a porous material might be modeled as a three-dimensional **n x n x n** grid of cells. The barriers separating adjacent pairs of cells might then be removed virtually with some probability **p** and remain with probability 1 **− p**. Simulating such a system is another application of union-find structures.

List-based Implementation

- Each set is stored in a sequence represented with a linked-list
- Each node should store an object containing the element and a reference to the set name

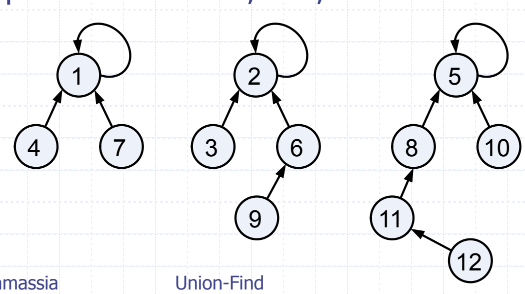


Analysis of List-based Representation

- When doing a union, always move elements from the smaller set to the larger set
  - Each time an element is moved it goes to a set of size at least double its old set
  - Thus, an element can be moved at most O(log n) times
- Total time needed to do n unions and m finds is O(n log n + m).

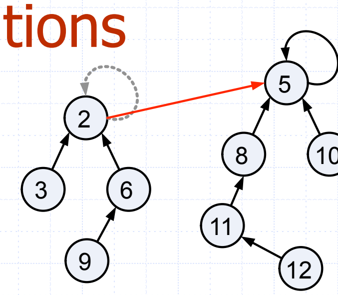
Tree-based Implementation

- Each element is stored in a node, which contains a pointer to a set name
- A node v whose set pointer points back to v is also a set name
- Each set is a tree, rooted at a node with a self-referencing set pointer
- For example: The sets "1", "2", and "5":

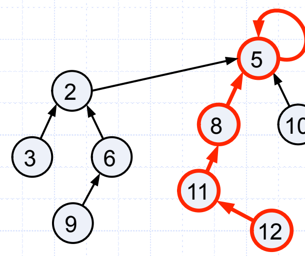


Union-Find Operations

- To do a union, simply make the root of one tree point to the root of the other

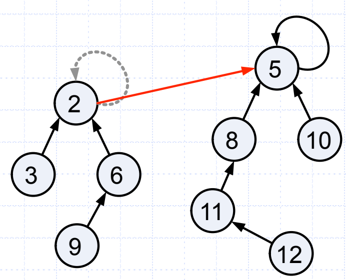


- To do a find, follow set-name pointers from the starting node until reaching a node whose set-name pointer refers back to itself



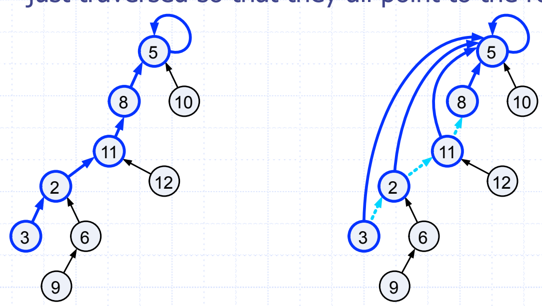
Union-Find Heuristic 1

- **Union by size**:
  - When performing a union, make the root of smaller tree point to the root of the larger
- Implies O(n log n) time for performing n union-find operations:
  - Each time we follow a pointer, we are going to a subtree of size at least double the size of the previous subtree
  - Thus, we will follow at most O(log n) pointers for any find.



Union-Find Heuristic 2

- **Path compression**:
  - After performing a find, compress all the pointers on the path just traversed so that they all point to the root



- Implies a fast "almost linear" time for n union-find operations.

Ackermann Function

The version of the Ackermann function we use is based on an indexed function, A_i, which is defined as follows, for integers $x \geq 0$ and $i > 0$:

$$A_0(x) = x + 1$$
$$A_{i+1}(x) = A_i^{(x)}(x),$$

where $f^{(k)}$ denotes the k-fold composition of the function f with itself. That is,

$$f^{(0)}(x) = x$$
$$f^{(k)}(x) = f(f^{(k-1)}(x)).$$

So, in other words, $A_{i+1}(x)$ involves making x applications of the A_i function on itself, starting with x. This indexed function actually defines a progression of functions, with each function growing much faster than the previous one:

- $A_0(x) = x + 1$, which is the increment-by-one function
- $A_1(x) = 2x$, which is the multiply-by-two function
- $A_2(x) = x2^x \geq 2^x$, which is the power-of-two function
- $A_3(x) \geq 2^{2^{\cdot^{\cdot^{\cdot^2}}}}$ (with x number of 2's), which is the tower-of-twos function
- $A_4(x)$ is greater than or equal to the tower-of-tower-of-twos function
- and so on.

Ackermann Function

We then define the **Ackermann function** as

$$A(x) = A_x(2),$$

which is an incredibly fast-growing function.

- To get some perspective, note that A(3) = 2048 and A(4) is greater than or equal to a tower of 2048 twos, which is much larger than the number of subatomic particles in the universe.

Likewise, its inverse, which is pronounced "alpha of n",

$$\alpha(n) = \min\{x: A(x) \geq n\},$$

is an incredibly slow-growing function. Even though $\alpha(n)$ is indeed growing as n goes to infinity, for all practical purposes, $\alpha(n) \leq 4$.

Fast Amortized Time Analysis

- For each node v in the union tree that is a root
  - define n(v) to be the size of the subtree rooted at v (including v)
  - identified a set with the root of its associated tree.
- We update the size field of v each time a set is unioned into v. Thus, if v is not a root, then n(v) is the largest the subtree rooted at v can be, which occurs just before we union v into some other node whose size is at least as large as v 's.
- For any node v, then, define the rank of v, which we denote as r(v), as $r(v) = [\log n(v)] + 2$:
- Thus, $n(v) \geq 2^{r(v)-2}$.
- Also, since there are at most n nodes in the tree of v, $r(v) \leq [\log n]+2$, for each node v.

Amortized Time Analysis (2)

- For each node v with parent w:
  - r (v) < r (w)

Proof: We make v point to w only if the size of w before the union is at least as large as the size of v. Let $n(w)$ denote the size of w before the union and let $n'(w)$ denote the size of w after the union. Thus, after the union we get

$$\begin{aligned} r(v) &= \lfloor \log n(v) \rfloor + 2 \\ &< \lfloor \log n(v) + 1 \rfloor + 2 \\ &= \lfloor \log 2n(v) \rfloor + 2 \\ &\le \lfloor \log(n(v) + n(w)) \rfloor + 2 \\ &= \lfloor \log n'(w) \rfloor + 2 \\ &\le r(w). \end{aligned}$$

■

- Thus, ranks are strictly increasing as we follow parent pointers.

Amortized Time Analysis (3)

- Claim: There are at most $n/ 2^{s-2}$ nodes of rank s.
- Proof:
  - Since r(v) < r(w), for any node v with parent w, ranks are monotonically increasing as we follow parent pointers up any tree.
  - Thus, if r(v) = r(w) for two nodes v and w, then the nodes counted in n(v) must be separate and distinct from the nodes counted in n(w).
  - If a node v is of rank s, then $n(v) \ge 2^{s-2}$.
  - Therefore, since there are at most n nodes total, there can be at most $n/2^{s-2}$ that are of rank s.

Amortized Time Analysis (4)

For the sake of our amortized analysis, let us define a ***labeling function***, $L(v)$, for each node v, which changes over the course of the execution of the operations in σ. In particular, at each step t in the sequence σ, define $L(v)$ as follows:

$$L(v) \;=\; \text{the largest } i \text{ for which } r(p(v)) \geq A_i(r(v)).$$

Note that if v has a parent, then $L(v)$ is well-defined and is at least 0, since

$$r(p(v)) \;\geq\; r(v) + 1 \;=\; A_0(r(v)),$$

because ranks are strictly increasing as we go up the tree U. Also, for $n \geq 5$, the maximum value for $L(v)$ is $\alpha(n) - 1$, since, if $L(v) = i$, then

$$\begin{aligned} n &> \lfloor \log n \rfloor + 2 \\ &\geq r(p(v)) \\ &\geq A_i(r(v)) \\ &\geq A_i(2). \end{aligned}$$

Or, put another way,

$$L(v) \;<\; \alpha(n),$$

for all v and t.

Amortized Time Analysis (5)

- Let v be a node along a path, P, in the union tree. Charge 1 cyber-dollar for following the parent pointer for v during a find:
  - If v has an ancestor w in P such that L(v) = L(w), at this point in time, then we charge 1 cyber-dollar to v itself.
  - If v has no such ancestor, then we charge 1 cyber-dollar to this find.
- Since there are most $\alpha(n)$ rank groups, this rule guarantees that any find operation is charged at most $\alpha(n)$ cyber-dollars.

Amortized Time Analysis (6)

- After we charge a node v then v will get a new parent, which is a node higher up in v's tree.
- The rank of v's new parent will be greater than the rank of v's old parent w.
- Any node v can be charged at most r(v) cyber-dollars before v goes to a higher label group.
- Since L(v) can increase at most α(n)-1 times, this means that each vertex is charged at most r(n)α(n) cyber-dollars.

Amortized Time Analysis (7)

- Combining this fact with the bound on the number of nodes of each rank, this means there are at most

$$s\,\alpha(n)\frac{n}{2^{s-2}} = n\,\alpha(n)\frac{s}{2^{s-2}}$$

cyber-dollars charged to all the vertices of rank s.

- Summer over all possible ranks, the total number of cyber-dollars charged to all nodes is at most

$$\sum_{s=0}^{\lfloor \log n \rfloor + 2} n\,\alpha(n)\frac{s}{2^{s-2}} \leq \sum_{s=0}^{\infty} n\,\alpha(n)\frac{s}{2^{s-2}} = n\,\alpha(n)\sum_{s=0}^{\infty}\frac{s}{2^{s-2}} \leq 8n\,\alpha(n),$$

so the total time for m union-find operations, starting with n singleton sets is O((n+m)α(n)).

