On Maximum Independent Set of Faces of Triangulated 2-Manifolds

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Abstract

This algorithm will work only for genus 0 objects. If you have any comments on this document, please do not hesitate to contact the authors. Thank you.

We prove bounds and an algorithm to approximate the maximum independent set of faces of a triangulated genus 0 manifold (where two faces are said to be adjacent if they share an edge). This algorithm uses $O(|V|)$ space, and runs in $O(|V||V'| + |V'|^3)$ time, where V is the vertex set of the triangulation of the manifold, and $V' \subset V$ is the set of all odd degree vertices.

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1 Introduction

The minimum vertex cover problem (VC) is one of the earliest computational problems known to be NP-complete [Karp 1972]: Given a graph $G = (V, E)$, find the smallest set of vertices $V' \in V$ such that every vertex $u \in V$ is either in V' or adjacent to another vertex $v \in V'$. The VC problem can be trivially reduced to the maximum independent set problem (IS): Given a graph $G = (V, E)$, find the largest set of vertices V'' such that no two vertices $u, v \in V''$ are adjacent to each other. These problems are NP-complete even in the substantially simplified case of 3-regular graphs [Garey et al. 1974].

In this paper, we present a deterministic method to compute a maximal independent set (conversely, a minimal vertex cover) of the dual graph of a triangulated manifold in polinomial time. The algorithm is shown to be correct, and this proof is used to demonstrate the bounds on the output size. Our method exploits the fact that many commonly used triangulated manifolds contain only a small number of odd-degree vertices. Finally, some practical examples are shown, along with relevant measurements that illustrate the bounds.

2 Theorems and Proofs

Before continuing, we must define some terms that will be used in the remainder of this paper.

Definition 1. *A* triangulated manifold *is a triangle mesh in which every edge is shared by exactly two triangles.*

Definition 2. *The* degree *of a vertex v is the number of triangles incident on v.*

Observation 1. *The number of odd-degree vertices present in a triangulated manifold M is even.*

Proof: In a triangulated manifold, number of edges around a vertex is same as the number of faces around a vertex. Hence the degree of a vertex is same as the number of edges incident on it. Each edge in M is adjacent to two mesh vertices, which implies that the sum of the degrees of all the vertices in M is $2E$, where E is the number of edges in M. Since the sum of the degrees of all vertices is even, the number of vertices with odd degree has to be even. □

Theorem 1. *The faces of a connected region P of a triangulated manifold M, that is homeomorphic to a disc with no odd-degree internal vertex can be 2-colored.*

Proof: We will prove this by induction. A single even-degree vertex u from M and its surrounding triangles is a connected patch that meets the conditions above. Its ring of faces can be trivially 2-colored by coloring the triangles alternatively red and black. Consider a patch P as defined above, that is 2-colored. Now choose an arbitrary boundary vertex v of P that has even degree in M. Add all the triangles incident on v to P. The newly added triangles can be trivially 2-colored by continuing the red-black pattern induced by the already colored triangles of P that are incident on v. The new, larger patch is still homeomorphic to a disk and has only even degree vertices in the interior. This patch can be extended indefinitely by adding new even-degree vertices that lie on its boundary. □

Definition 3. *A closed boundary of a patch that has even (or odd) number of edges and even (or odd resp.) number of odd degree vertices along it, is called a colorable boundary.*

Theorem 2. *A connected patch P with no odd-degree internal vertex and an internal colorable boundary can be 2-colored.*

Proof: Let the total number of faces incident on a even degree vertices along the internal boundary be f_e. Let the total number of faces incident on a odd degree vertices along the internal boundary be f_o. Let the number of edges along the internal boundary be e_b. We know that f_e, f_o, and e_b are all even. Total number of faces incident on all the vertices on the internal boundary equals to $(f_e + f_o - e_b)$ which is even. Hence these faces can be 2-colored. Once they are colored, the internal boundary along with the 1-ring of faces can be considered as a consistently colored patch B, to which other vertices of P are added to grow the patch B to become P. Since the added vertices are even-degree vertices, the consistency of the 2-coloring can be maintained as proved by Theorem 1. □

Lemma 1. *A connected patch P with no odd-degree internal vertex and any number of internal colorable boundaries can be 2-colored.*

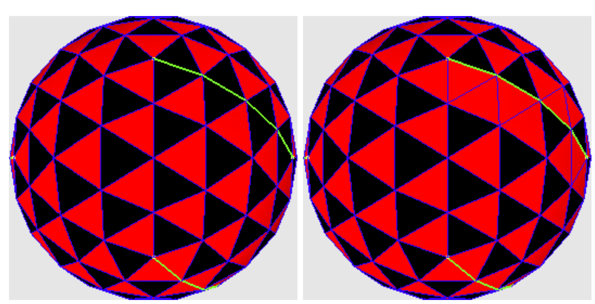

Figure 1. (Left) The two coloring with the shortest edges between the odd-degree vertices being considered as boundary edges. (Right) Adjacent faces that are chosen (colored black) are fixed along the path.

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Dataset	$\|F\|$	$\|E'\|$	$\frac{\|F\|-\|E'\|}{2}$	Computed IS size
Sphere	320	24	148	148
Triceratops	5660	836	2412	2415
Fandisk	12946	697	6124.5	6135
Torus	9600	0	4800	4800
2-torus	28288	96	14096	14104

Table 1. Predicted lower bounds and actual size of the computed maximum independent set for the dual graphs of several triangulated manifolds.

3 Finding Maximal Independent Set of Faces

Let E, V and F be the set of edges, vertices, and faces of the triangulated two manifold M. Let $V' \subset V$ be the set of all odd-degree vertices of M. Let $E(v'_i, v'_j) \subset E$ be the sequence of edges forming the shortest path between $v'_i \in V'$ and $v'_j \in V'$.

1. Construct a weighted, undirected, complete graph $G' = (V', E_{odd})$ with $|E(v'_i, v'_j)|$ as the weight of the edge between v'_i and v'_j. Since $|V'|$ is even, and G' is a complete graph, the minimum weight perfect matching of G', named $m(G')$, exists.

2. Let $E' = \bigcup_{E_{odd}(v'_i, v'_j) \in m(G')} E(v'_i, v'_j)$. In the input manifold M, edges in E' are tagged as hole boundaries for the next step. In essence, we cut the manifold along the edges in E'. Note that the internal boundaries thus formed have even number of edges and even number of odd degree vertices on them. Further, all vertices of V' are in the boundaries of M, and all internal vertices of M are of even degree. Hence a 2-coloring exists in M.

3. An arbitrary triangle $t \in M$ is chosen and colored black. Starting from t, the manifold is flooded, tagging each triangle alternatively as red or black. Note that depending on the color of t, the result will be almost, but may not be completely complementary. We consider triangles (or dual nodes) colored black as candidates to belong in the maximum independent set of the given manifold with internal boundaries.

4. As a final step, we stitch back the cut edges of E'. After stitching, if two black triangles become adjacent to each other, one of them is colored red (removed from the independent set). Number of such re-coloring (k) can be at most $|E'|$ since such re-coloring happens only for the triangles incident on E'. If $k > |\frac{E'}{2}|$, then we switch the initial coloring of t to red and repeat the process. Now the new number of re-recolorings $k' = |E'| - k$.

The absolute maximum size of any independent set on the dual graph of a triangulated manifold is $\frac{F}{2}$. As mentioned above, depending on the color chosen for the initial triangle t, this overhead of re-coloring can be k or $|E'| - k$. We choose the color for t that produces the smaller overhead, and attain a final worst-case lower bound of $\frac{|F|-|E'|}{2}$.

4 Results and Conclusion

The complexity of the shortest path algorithm is $O(|V||V'|)$ by simple flooding of hop distance from every source vertex in V'. The complexity of the weighted perfect matching is $O(|V'|^3)$. Finally, the coloring is a linear time algorithm on number of faces. Hence the total complexity is $O(|V||V'| + |V'|^3)$. Table 1, the performance of our method is very close to the stated bounds, which suggests a close proximity to the optimal value. These results can be extended to manifolds with boundaries also.

References

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