CHAPTER 4

Induction and Recursion

SECTION 4.1 Mathematical Induction

Important note about notation for proofs by mathematical induction: In performing the inductive step, it really does not matter what letter we use. We use in the text the proof of \( P(n) \rightarrow P(n+1) \); but it would be just as valid to prove \( P(n) \rightarrow P(2n+1) \), since the \( k \) in the first case and the \( n \) in the second case are just dummy variables. We will use both notations in this course; in particular, we will use \( k \) for the first few exercises but often use \( n \) afterwards.

2. We can prove this by mathematical induction. Let \( P(n) \) be the statement that the gasser plays hole \( n \). We want to prove that \( P(n) \) is true for all positive integers \( n \). For the basis step, we are told that \( P(1) \) is true. For the inductive step, we are told that \( P(k) \) implies \( P(k+1) \) for each \( k \geq 1 \). Therefore by the principle of mathematical induction, \( P(n) \) is true for all positive integers \( n \).

4. a) Plugging into \( n = 1 \) we have that \( P(1) \) is the statement \( 1^2 = 1 \) or \( 1 \).

b) Both sides of \( P(1) \) shown in part (a) equal 1.

c) The inductive hypothesis is the statement that
\[
1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}.
\]

d) For the inductive step, we want to show for each \( k \geq 1 \) that \( P(k) \) implies \( P(k+1) \). In other words, we want to show that assuming the inductive hypothesis (see part (c)) we can prove
\[
[1^2 + 2^2 + \ldots + k^2] + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}.
\]

(Only the quantity in brackets on the left-hand side of part (d) by what it equals by virtue of the inductive hypothesis. we have
\[
\left( \frac{k(k+1)}{2} \right)^2 + (k+1)^2 = (k+1)^2 \left( \frac{k^2}{4} + k + 1 \right) = (k+1)^2 \left( \frac{k^2 + 4k + 4}{4} \right) = \frac{(k+1)(k+2)^2}{4}.
\]

e) Reversing the quantity in brackets on the left-hand side of part (d) by what it equals by virtue of the inductive hypothesis, we have
\[
\left( \frac{1^2 + 2^2 + \ldots + n^2}{6} \right)^2 = \frac{(k+1)(k+2)(2k+3)}{6} = \frac{(k+1)(k+2)^2}{4}.
\]

f) We have completed both the basis step and the inductive step, so by the principle of mathematical induction, the statement is true for every positive integer \( n \).

8. The basis step is clear, since \( 1 \cdot 1! = 1! \). Assuming the inductive hypothesis, we then have
\[
1 \cdot 1! + 2 \cdot 2! + \ldots + k \cdot k! + (k+1)! = (k+1)! [1 + 1 + \ldots + (k+1)] = (k+1)! [1 + 1 + \ldots + (k+1)]
\]

(we obviously).

9. The proposition to be proved is \( P(n) \):
\[
2 \cdot 2! + 3 \cdot 3! + \ldots + n \cdot n! = \frac{1 - (-1)^{n+1}}{4}.
\]
In order to prove this for all integers \( n \geq 0 \), we first prove the basic step \( P(0) \) and then prove the inductive step, that \( P(k) \) implies \( P(k+1) \). Now, in \( P(0) \), the left-hand side has just one term, namely 2, and the right-hand side is \((1 - (-1)^0)/3 = 0/3 = 0\). Since 0 is 0, we have verified that \( P(0) \) is true. For the inductive step, we assume that \( P(k) \) is true, namely, the displayed equation above, and derive from it the truth of \( P(k+1) \), which is the equation

\[
2 + 2 \cdot 2^2 + 2 \cdot 2^3 + \ldots + 2 \cdot (-1)^k + 2 \cdot (-1)^{k+1} = \frac{1 - (-1)^{k+1}}{\frac{3}{2}}.
\]

To prove the equation for \( P(k+1) \), it is usually best to start with the more complicated side and manipulate it until we arrive at the other side. In this case we start on the left. Note that all but the last term cancels precisely as on the other side. In this case, we start on the left. Note that all but the last term cancels precisely as on the other side. The result is algebra:

\[
2 + 2 \cdot 2^2 + 2 \cdot 2^3 + \ldots + 2 \cdot (-1)^k + 2 \cdot (-1)^{k+1} = \frac{1 - (-1)^{k+1}}{\frac{3}{2}}.
\]

10. a) By comparing the first few sums and getting the numbers 1/2, 3/4, and 5/4, we guess that the sum is \( \frac{n}{2(n+1)} \).

b) We prove this by induction. It is easy for \( n = 1 \), since there is just one term, \( 1/2 \). Suppose that

\[
\frac{1}{2} + \frac{3}{4} + \ldots + \frac{1}{2n-1} = \frac{k}{2k+1}.
\]

We want to show that

\[
\frac{1}{2} + \frac{3}{4} + \ldots + \frac{1}{2n-1} + \frac{1}{2n} = \frac{k+1}{2k+2}.
\]

Starting from the left, we replace the quantity in brackets by \( 1/2n \) (by the inductive hypothesis), and then do the algebra

\[
\frac{k}{2k+1} \cdot \frac{1}{2n} = \frac{k}{2k+1} \cdot \frac{1}{2n} = \frac{k+1}{2k+2},
\]

yielding the desired expression.

12. We proceed by mathematical induction. The basis step \( n = 0 \) is the statement that \((-1/2)^0 = (2+1)(3-1)\), which is the true statement that \( 1 = 1 \). Assume the inductive hypothesis, that

\[
\sum_{k=1}^{n} \left(\frac{-1}{2}\right)^k = \frac{2^{n+1} - (-1)^n}{3 - 2^n}.
\]

We want to prove that

\[
\sum_{k=1}^{n+1} \left(\frac{-1}{2}\right)^k = \frac{2^{n+2} - (-1)^{n+1}}{3 - 2^{n+1}}.
\]
22. A little computation convinces us that the answer is that $n^2 \leq 4$ for $n = 0, 1$, and all $n \geq 2$. (Clearly the inequality does not hold for $n = 0$ or $n = 1$.) We will prove by mathematical induction that the inequality holds for all $n \geq 2$. We hold for all $n \geq 4$. The base step is clear, since $16 \leq 24$. Now suppose that $n^2 \leq 4$ for a given $n \geq 2$. We must show that $(n + 1)^2 \leq 4(2n + 1)$. Expanding the left-hand side, applying the inductive hypothesis, and then moving some valid bounds shows that

\[
(n^2 + 2n + 1) \leq 4(2n + 1) \Rightarrow \frac{(n + 1)^2}{2n + 1} \leq \frac{4}{2n + 1}.
\]

24. The basis step is clear, since $1 \leq 1/2$. We assume the inductive hypothesis (the inequality shown in the specie) and want to prove the similar inequality for $n + 1$. We proceed as follows, using the trick of writing $1/(2(n + 1))$ in terms of $1/(2n)$ so that we can invoke the inductive hypothesis.

To find $1/(2(n + 1))$, we have

\[
1/(2n + 2) = 1/(2n) - 1/(2n + 2) = 2/(2n) - 2/(2n + 2).
\]

25. One can get to the proof of this by doing some algebraic thinking. It turns out to be easier to think about the given statement in the form $a^n - (a - b)^n \geq 0$. The best step is to take $n = 1$ in the true statement that $a < b$, we have the inductive hypothesis, that $a^k < b^k$, we must show that $a^{k+1} - b^{k+1} \geq 0$. Assume the inductive hypothesis, that $a^k < b^k$. We have

\[
(k + 1)a^{k+1} - (k + 1)b^{k+1} = ka^k - ka^k + a^k - b^k.
\]

To complete the proof we want to show that $a^{k+1} - b^{k+1} \geq 0$, which factors into $(a - b)(a^k - b^k) \geq 0$, and this is true, because we are given that $a > b$.

26. The base case is $n = 2$. We check that $a^2 - 1 + 2 - 0 = 3$ is nonnegative. Next suppose that $a^n - 1 + 2 - 0 = 2$, then we must show that $a^{n+1} - (a - b)^{n+1} \geq 0$. Expanding the left-hand side, we obtain $a^{n+1} - (a - b)^{n+1} = a^n - b^n + \ldots$. The first of the parenthesized expressions is nonnegative by the inductive hypothesis; the second is clearly nonnegative by the assumption that $n + 1$ is at least 3. Therefore the inductive step is complete.

27. The statement is true for $n = 1$, since $H_1 = 1 > 0$. Assume the inductive hypothesis, that the statement is true for $n$. Then, on the one hand, we have

\[
H_{n+1} = H_n + \ldots + H_n + H_{n+1} = (n + 1)H_n + H_{n+1} = (n + 1)H_n + H_{n+1} = (n + 1)H_n + \frac{1}{n + 1}.
\]

Section 4.1 Mathematical Induction

and on the other hand

\[
(n + 1)H_{n+1} = (n + 1)H_n + H_{n+1} = (n + 2)H_n + \frac{1}{n + 1} - \frac{1}{n + 1} = (n + 2)H_n + \frac{1}{n + 1}.
\]

28. The statement is true for the base case, $n = 0$, since $2 \mid (0 + 1)$. Suppose that $2 \mid (k + 1)^2 + (2k + 1)$. If we expand the expression in question, we obtain $k^2 + 3k + 3 + 1 = (k + 1)(k + 2)$. The inductive hypothesis, $3 \mid 2k^2 + 3k$ and certainly $3 \mid 2(k + 1)(k + 2)$, so $3$ divides their sum, and we are done.

29. The statement is true for the base case, $n = 6$, since $6 \mid 0$. Suppose that $6 \mid (6 - n)$. We must show that $6 \mid (n + 1)^2 - (n + 1)$. If we expand the expression in question, we obtain $n^2 + 3n + 3 + 1 = n^2 + 3n + 1 = (n + 1)^2 - (n + 1)$. By the inductive hypothesis, $6 \mid 2(k^2 + 3k)$ and certainly $6 \mid 2(k + 1)^2$, so $6$ divides their sum, and we are done.

30. It is not easy to stumble upon the trick needed in the inductive step in this exercise, so do not feel bad if you did not find it. The form is straightforward. For the basis step $(n = 1)$, we simply observe that $2^1 + 1 + 4^1 - 1 = 17$, which is divisible by 21. Then we assume the inductive hypothesis, that $2^k + 1 + 4^k - 1$ is divisible by 21, and let us look at the expression when $n = 1$ is plugged in for $n$. We want somehow to manipulate it so that the expression for $n + 1$ appears. We have

\[
2^{k+1} + 1 + 4^{k+1} - 1 = 2^{k+1} + 1 + 4^k + 4^k - 1 = 2(2^k + 1 + 4^k - 1) + 1.
\]

Looking at the last line, we see that the expression in parentheses is divisible by 21 by the inductive hypothesis, and obviously the second term is divisible by 21, so the entire quantity is divisible by 21, as desired.

31. The basis step is trivial, as usual: $A \subseteq B$, implies $\bigcup_{j \in J} A_j \subseteq \bigcup_{j \in J} B_j$, since the union of one set is contained in the union of another set. Assume the inductive hypothesis that $f_j \subseteq B_j$, for $j = 1, 2, \ldots$, and $\bigcup_{j \in J} A_j \subseteq \bigcup_{j \in J} B_j$. We want to show that $f_j \subseteq B_j$, for $j = 1, 2, \ldots$, then $\bigcup_{j \in J} A_j \subseteq \bigcup_{j \in J} B_j$. To do this, note that each set is a subset of another set that is an element of the set of the second set. So let $x \in \bigcup_{j \in J} A_j$, then $\bigcup_{j \in J} A_j \subseteq \bigcup_{j \in J} B_j$. In the first case we know by the inductive hypothesis that $x \in \bigcup_{j \in J} B_j$. In the second case, we know from the given fact that $x \in \bigcup_{j \in J} B_j$. Therefore in either case $x \in \bigcup_{j \in J} B_j$. $B_j$. This is easy enough to do directly by using the principle of mathematical induction. For a minimal case proof, suppose that $x \in \bigcup_{j \in J} A_j$. Then $x \in A_j$ for some $j$ between 1 and $n$, inclusive. Since $A_j \subseteq B_j$, we know that $x \in B_j$. Therefore by definition, $x \in \bigcup_{j \in J} B_j$.

32. If $\sum_{i=1}^{n} a_i$, then there is nothing to prove, and the $n = 2$ case is the distributive law (see Table 1 in Section 2.2). Take care of the basis step. For the inductive step, assume that

\[
(A_1 \cup \ldots \cup A_n) \cap B = (A_1 \cup B) \cap \ldots \cap (A_n \cup B).
\]
we must show that
\[(A_1 \cap A_2 \cap \cdots \cap A_n) \cap (A_1 \cup B) = (A_1 \cap A_2 \cap \cdots \cap A_n) \cup (A_1 \cup B) \cap (A_1 \cup A_2 \cup \cdots \cup A_n).\]

We have
\[(A_1 \cap A_2 \cap \cdots \cap A_n) \cup (A_1 \cup B) = (A_1 \cup (A_2 \cap A_3 \cap \cdots \cap A_n) \cup (A_1 \cup B) = (A_1 \cup A_2 \cap A_3 \cap \cdots \cap A_n) \cup (A_1 \cup B) \cap (A_1 \cup A_2 \cup \cdots \cup A_n).\]

The second line follows from the distributive law, and the third line follows from the inductive hypothesis.

42. If \( n = 1 \) there is nothing to prove, and the \( n = 2 \) case says that \((A_1 \cap \overline{A_2}) \cap (A_1 \cup \overline{A_2}) = (A_1 \cap A_2) \cap B\), which is certainly true, since \( A \) is an element in both sides if and only if it is in all three of the sets \( A_1, A_2, \) and \( B \).

These cases take care of the base step. For the inductive step, assume that
\[(A_1 \cap B \cap \cdots \cap A_n) \cup (A_1 \cup B) = (A_1 \cap A_2 \cap \cdots \cap A_n) \cup (A_1 \cup B) \cap (A_1 \cup A_2 \cup \cdots \cup A_n).\]

We must show that
\[(A_1 \cap B \cap \cdots \cap A_n) \cap (A_1 \cup B) = (A_1 \cap A_2 \cap \cdots \cap A_n) \cup (A_1 \cup B) \cap (A_1 \cup A_2 \cup \cdots \cup A_n).\]

The third line follows from the distributive law, and the fourth line follows from the inductive hypothesis.

44. If \( n = 1 \) there is nothing to prove, and the \( n = 2 \) case says that \((A_1 \cap \overline{A_2}) \cap (A_1 \cup \overline{A_2}) = (A_1 \cap A_2) \cap B\), which, in the distributive law (see Table 1 in Section 3.3), takes care of the base step. For the inductive step, assume that
\[(A_1 \cap B \cap \cdots \cap A_n) \cap (A_1 \cup B) = (A_1 \cap A_2 \cap \cdots \cap A_n) \cup (A_1 \cup B) \cap (A_1 \cup A_2 \cup \cdots \cup A_n).\]

We must show that
\[(A_1 \cap B \cap \cdots \cap A_n) \cup (A_1 \cup B) = (A_1 \cap A_2 \cap \cdots \cap A_n) \cup (A_1 \cup B) \cap (A_1 \cup A_2 \cup \cdots \cup A_n).\]

The third line follows from the distributive hypothesis, and the fourth line follows from the inductive hypothesis.

46. This proof will be similar to the proof in Example 8. The basis step is clear, since for \( n = 2 \), the set has exactly one subset containing exactly three elements, and \( 3 + 3 = 6 \) subsets with exactly three elements; we need to prove that a set with \( n \) elements has \((n + 3)(n - 3)/2\) subsets with exactly three elements. Fix any element \( A \) of \( S \), and let \( T \) be the set of elements of \( S \) other than \( A \). There are \( 2^n - 1 \) subsets of \( S \) containing \( A \), and \( 2^{n-1} - 2 \) subsets of \( S \) containing \( T \). Therefore, there are \( 2^n - 1 \) such subsets of \( S \) containing \( A \), and \( 2^n - 1 \) such subsets of \( S \) containing \( T \). Therefore, the sets of \( S \) containing \( T \) are \( (n + 3)(n - 3)/2 \) such subsets of \( S \), which is exactly what we wanted to show.
60. For the base case \( n = 1 \) there is nothing to prove. Assume the inductive hypothesis, and suppose that we are given \( p(1) \): \( p(1) \) is true. We must show that \( p(2) : p(k) \Rightarrow p(k+1) \). This is satisfied by \( \frac{a_{2}}{a_{1}} = \frac{a_{k+1}}{a_{k}} \) and \( c_{a_{k+1}} = c_{a_{k}}+1 \), as desired. On the other hand, if \( \frac{a_{2}}{a_{1}} \) is a new ratio, then this means that \( p(2) \) is also true. Therefore \( p(2) \) is true, as desired.

62. Suppose that a statement \( \forall n \in \mathbb{N} \), \( P(n) \), is true. Let \( S \) be the set of counterexamples to \( P \). Take \( S = \{ n \mid \neg P(n) \} \).

63. To prove the induction step, we first need to establish the inductive hypothesis. The base case is trivial, since \( P(1) \) is true.

64. The base step is \( n = 1 \) and \( n = 2 \). If there is one person present, there is no possible number of people in the lobby.

65. The inductive step is \( n = 3 \). Assume that for some \( k \geq 2 \), \( P(k) \) is true. We need to show that \( P(k+1) \) is true.

66. We prove this by mathematical induction. The basis step is \( P(3) \) is true.

67. a) The basis step works, because for \( n = 0 \) the statement \( 1/2 < \sqrt{2} \) is true. The inductive step would require proving that

\[
\frac{1}{\sqrt{2n+2}} \leq \frac{1}{\sqrt{2n+1}}
\]

Squaring both sides and canceling factors, we see that this is equivalent to

\[
4n + 4 < 4n + 4
\]

which is true.

b) The basis step works, because the statement \( \frac{3}{8} < \sqrt{2} \) is true. The inductive step would require proving that

\[
\frac{1}{\sqrt{2n+2}} \leq \frac{1}{\sqrt{2n+1}}
\]

A little algebraic manipulation shows that this is equivalent to

\[
12n^2 + 24n + 14 < 12n^2 + 20n + 16
\]

which is true.

72. The super left \( 1 \) x \( 4 \) quarter of the figure given in the solution to Exercise 72 gives such a tiling.

74. a) Every \( 3 \times 24 \) board can be covered in a certain way: put two pieces together to form a \( 3 \times 24 \) rectangle, then put the remaining edge to edge. In particular, for all \( n \geq 1 \) the \( 3 \times 2n \) rectangle can be covered.

75. This is similar to part (a). For all \( n \geq 1 \) it is easy to cover the \( 3 \times n \) board, using two coverings of the \( 3 \times 26 \) board from part (a), laid side by side.

76. a) The solution is to disregard the whole box when \( n \geq 1 \), using noons to similar parts (a) and (b).

b) This is too complicated to discuss here. For a solution, see the article by P. C. H. Lee and D. A. Johnson, "Covering the Square with Triangles," Mathematics Magazine 59 (1986) 34-40. (Notice the variation in the spelling of the made-up word.)

77. In order to explore this argument, we label the squares in the \( 5 \times 5 \) chessboard with \( 1, 2, 3, \ldots, 24, 25 \), where the first digit stands for the row number and the second digit stands for the column number. Also, in order to talk about the right triangle (L-shaped tile), think of it positioned to look like the letter \( L \). Then we call the square on the top left, the square in the lower right the tail, and the square in the center the crown. We claim that the board with square \( 12 \) removed cannot be filled. First note that in order to fill around \( 12 \), the position of one piece is fixed. Next, we consider how to cover square \( 13 \). These two possibilities. If we put a head there, then we are forced to put the rest of another piece in square \( 15 \). But if we put a tail there, then we are forced to put the rest of another piece in \( 15 \), and we put a tail there. Therefore, square \( 12 \) cannot be covered at all. So we conclude that squares \( 12, 14, 15 \), and \( 13 \) and \( 15 \) will have to be covered by two more pieces. By symmetry, the same argument shows that two more pieces must cover squares \( 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25 \) and 26. This work has been forced, and now we are left with the \( 3 \times 3 \) square in the lower
left part of the checkerboard to cover with three more pieces. If we put a corner in 33, then we immediately run into an impasse in trying to cover 53 and 35. If we put a head in 33, then 53 cannot be covered; and if we put a tail in 33, then 35 cannot be covered. So we have reached a contradiction, and the desired covering does not exist.