b) We apply the definition:

\[ a(0) = 0 \]
\[ a(1) = a(a(0)) = 1 - a(a(0)) = 1 - 0 = 1 \]
\[ a(2) = 2 - a(a(1)) = 2 - 1 = 1 \]
\[ a(3) = 3 - a(a(2)) = 3 - 1 = 2 \]
\[ a(4) = 4 - a(a(3)) = 4 - 2 = 2 \]
\[ a(5) = 5 - a(a(4)) = 5 - 2 = 3 \]
\[ a(6) = 6 - a(a(5)) = 6 - 3 = 3 \]
\[ a(7) = 7 - a(a(6)) = 7 - 3 = 4 \]

Thus, \[ a(7) = 7 - a(a(6)) = 7 - 3 = 4 \]

We apply the definition:

\[ b(1) = 1 \]
\[ b(2) = 2 - a(a(1)) = 2 - 1 = 1 \]
\[ b(3) = 3 - a(a(2)) = 3 - 1 = 2 \]
\[ b(4) = 4 - a(a(3)) = 4 - 2 = 2 \]
\[ b(5) = 5 - a(a(4)) = 5 - 2 = 3 \]
\[ b(6) = 6 - a(a(5)) = 6 - 3 = 3 \]
\[ b(7) = 7 - a(a(6)) = 7 - 3 = 4 \]

Thus, \[ b(7) = 7 - a(a(6)) = 7 - 3 = 4 \]

72. The first term \( a_n \) tells how many \( 1 \)'s there are. If \( n \geq 2 \), then the sequence would not be nondecreasing.

SECTION 5.1 The Basics of Counting

2. By the product rule there are \( 2 \times 27 \times 999 \) offices.

4. By the product rule there are \( 12 \times 5 \times 72 \) different types of shirts.

6. By the product rule there are \( 4 \times 6 = 24 \) routes.

8. There are 26 choices for the first initial, then 26 choices for the second, if no letter is to be repeated, then 24 choices for the third. (We interpret "repeated" broadly, so that a string like 5599 0, for example, is prohibited, as well as a string like RRR.) Therefore by the product rule the answer is \( 26 	imes 26 	imes 24 = 15,600 \).

10. We have two choices for each bit, so there are \( 2^8 = 256 \) bit strings.

12. We use the same rule, adding the number of bit strings of each length up to 8. If we include the empty string, then we get \( 2^0 = 1 \) as before, \( 2^1 = 2 \) as before, \( 2^2 = 4 \) as before, \( 2^3 = 8 \) as before, \( 2^4 = 16 \) as before, \( 2^5 = 32 \) as before, \( 2^6 = 64 \) as before, \( 2^7 = 128 \) as before, \( 2^8 = 256 \) as before, which sums to 256 as expected. We use the formula for the sum of a geometric progression—see Example 4 in Section 4.1.

13. If \( n = 0 \), then the empty string—vacuously—satisfies the condition (or does not, depending on how one views it). If \( n = 1 \), then there is only one, namely the string 1. If \( n \geq 2 \), then such a string is determined by specifying the \( n - 2 \) bits between the first bit and the last. Here, there are \( 2^{n-2} \) such strings.

14. We can subtract from the number of strings of length 4 of lower case letters the number of strings of length 4 of lower case letters other than \( x \). Thus the answer is \( 26^4 - 25^4 = 60,664 \).

14. Because neither 5 nor 31 is divisible by either 3 or 4, whether the ranges are meant to be inclusive or exclusive of these endpoints is essential.

a) There are \( 31/4 = 7 \) integers less than 31 that are divisible by 4, and \( 5/31 = 1 \) of them is less than 3 as well. This leaves 19 - 1 = 19 numbers between 5 and 31 that are divisible by 3. They are 6, 9, 12, 15, 18, 21, 24, 27, and 30.

b) There are \( 24/4 = 6, 1 \) less than 24 that are divisible by 4, and \( 25/5 = 5 \) of them is less than 5 as well. This leaves 24 - 6 = 18 numbers between 5 and 24 that are divisible by 3. They are 6, 9, 12, 15, 18, 21, 24, 27, and 30.

15. A number is divisible by both 3 and 4 only if it is divisible by their least common multiple, which is 12. Obviously there are two such numbers between 5 and 31, namely 12 and 24. We also wish to rule out the previous two pairs in exactly what we are looking for here.
20. Every seventh number is divisible by 7. Therefore there are \(100 \times 1/7 = 14\) such numbers. Note that we use the floor function, because the 14th multiple of 7 does not occur until the number 14 has been reached.


22. It will be useful to note that there are exactly 10000 numbers in this sequence.

a) Every 10th number is divisible by 10, so the answer is simply 10000 - 1000 = 9000.

b) Every 5th number is also divisible by 5, so the answer is simply 9000 - 1000 = 8000.

c) Since the numbers are divisible by 4, the answer is simply 9000 - 1000 = 8000.

d) Since the numbers are divisible by 2, the answer is simply 9000 - 1000 = 8000.

e) Since the numbers are divisible by 1, the answer is simply 9000 - 1000 = 8000.
38. We know that there are \( 2^{10} \) Strips in all. Clearly 101 of them do not have more than one element, namely the empty set and the 100 sets consisting of 1 element. Therefore the answer is \( 2^{10} - 101 = 1,024 - 101 = 923 \).

40. a) We first place the bride in any of the 6 positions. Then, from left to right in the remaining positions, we choose the other five people to be in the picture, this can be done in \( 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 = 720 \) ways. Therefore the answer is \( 6 \cdot 720 = 4,320 \).

b) We first place the bride in any of the 6 positions, and then place the groom in any of the 5 remaining positions. Then, from left to right in the remaining positions, we choose the other four people to be in the positions. This can be done in \( 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \) ways. Therefore the answer is \( 6 \cdot (5 \cdot 4 \cdot 3 \cdot 2) = 720 \).

c) From part (b) there are 9,072 ways for the bride to be in the picture. There are \( 5 \) different ways for the bride to be in the picture. Therefore there are 9,072 \( \cdot 5 \) ways for both the bride and groom to be in the picture. Similarly there are 40,320 ways for just the groom to be in the picture. Therefore the answer is 40,320 \( \cdot \frac{5}{6} \) = 33,600.

43. There are \( 2^n \) strings that begin with two 0's (since there are two choices for each of the last five bits). Similarly there are \( 2^{n-1} \) strings that end with three 1's. Furthermore, there are \( 2^n \) strings that both begin with two 0's and end with three 1's (since only bits 3 and 4 are free to be chosen). By the inclusion-exclusion principle, there are \( 2^n + 2^{n-1} - 2 \) such strings all together.

44. First, we count the number of bit strings of length 10 that contain five consecutive 0's. We will base the count on the first bit, the last five bits, and the actual bit string of length 10 containing all 0's. If it starts in the first bit, then the first five bits are all 0's, but there are five choices for the last five bits, therefore there are \( 2^5 \cdot 2^5 \) strings. If it starts in the second bit, then the first bit must be 1, the next five bits are all 0's, but there are free choices for the last five bits, therefore there are \( 2^5 \cdot 2^5 \) strings. If it starts in the third bit, then the second bit must be 1, the next five bits are all 0's, but there are free choices for the last five bits, therefore there are \( 2^5 \cdot 2^5 \) strings. Similarly, if the first bit is 0 but the last five bits are arbitrary, then there are \( 2^5 \cdot 2^5 \) strings.

45. This is a straightforward application of the inclusion-exclusion principle. \( 38 + 27 - 7 = 58 \) (we need to subtract the 7 double major counts twice in the sum).

46. Under matters here, the 0-bits in the first 10/indexes, for example, are different than the 0-bits in the 0-bits in the second 10/indexes. By the same rule, we can add the number of bit strings with two, three, four, and five or more. By the product rule, these are \( 2^3 \), \( 2^2 \), \( 2^1 \), and the last, respectively, so the sum is \( 2^3 + 2^2 + 2^1 = 8 + 4 + 2 = 14 \).

50. We need to compute the number of variable names of length \( i \) for \( i = 1, 2, \ldots, 8 \), and add. A variable name of length \( i \) is specified by choosing a first character, which can be done in \( 53 \) ways (25 letters and 1 underscore, \( \sim 2 \) letters), and \( i - 1 \) other characters, each of which can be done in \( 36 + 10 \) ways: underscore to choose (0's), and 0-9 (10 others). Therefore the answer is \( \sum_{i=1}^{8} \left( 53 \cdot (36+10)^{i-1} \right) \).