Note Title 10/14/2013

Theorems of form If P then Q.

Direct: Assume P, prove that Q follows.
Contrapositive: Assume $\neg Q$ prove that $\neg P$ follows.

Theorem If $0 \le x \le 2$ then $-x^3 + 4x + 1 > 0$.

Scratch $x = 0$ $1 > 0$

$x = 1$: $-1 + 4 + 1 = 4 > 0$.

Factor $-x^3 + 4x = x(4 - x^2)$
$= x(2 - x)(2 + x)$

Proof: Assume $0 \le x \le 2$.

For x in this range:

x, $2 - x$, $2 + x$ are all ≥ 0.

Therefore $x(2-x)(2+x) \ge 0$

$x(2-x)(2+x) + 1 \ge 1 > 0$

$4x - x^3 + 1 > 0$

Proof by Contrapositive:

Thm If $x^3 + 3x^2 + 2x + 1 \leq 0$ then $x < 0$.

Proof by contrapositive. Assume $\boxed{x \geq 0}$

Then. $x^3 \geq 0$
$3x^2 \geq 0$
$2x \geq 0$

Therefore $+1$ $x^3 + 3x^2 + 2x \geq 0$ $+1$

$x^3 + 3x^2 + 2x + 1 \geq 1 > 0$

Therefore $x^3 + 3x^2 + 2x + 1 > 0$

For any integer n

$\Rightarrow$ If n is odd then 3n+8 is odd. Direct

$\not\Rightarrow$ If 3n+8 is odd then n is odd. Contrapositive.

Assume n is not odd. n is even.

There is some integer c such that $n = 2c$.

$3n + 8 = 3(2c) + 8 = 2[3c+4] \Rightarrow$ even

3c+4 is an integer

n is even iff $\exists c$ $n = 2c$
n is odd iff $\exists c$ $n = 2c+1$

If $\underbrace{3n+8 \text{ is odd}}_{P}$ then $\underbrace{n \text{ is odd}}_{Q}$.

Assume $\neg Q$ prove $\neg P$

Assume n is even prove $3n+8$ is even.

Theorem If $\overbrace{r \text{ is irrational}}^{P}$ then $\overbrace{\sqrt{r} \text{ is irrational}}^{Q}$.

a is rational if $\exists x,y$ $a = x/y$ x and y are integers.

b is irrational if it is not rational.

Assume $\neg Q$: $\sqrt{r}$ is not irrational $\equiv$ $\sqrt{r}$ is rational

Prove $\neg P$: r is not irrational $\equiv$ r is rational

Proof: $\sqrt{r}$ is rational so $\exists x,y$ such that x & y are integers and

$$\left(\sqrt{r}\right)^2 = \left(\frac{x}{y}\right)^2$$

$r = \frac{x^2}{y^2}$. Since x + y are integers, x^2 + y^2 are integers. Therefore r is rational. ☒

Theorem If the product of two positive real numbers is > 100 then at least one of them is greater than 10.

If x, y $x > 0$, $y > 0$ and $xy > 100$ then $x > 10$ or $y > 10$.

Assume x, y are positive real numbers.

and $\neg(x > 10 \text{ or } y > 10) \equiv \neg(x > 10) \text{ and } \neg(y > 10)$

$x \leq 10$ and $y \leq 10$.

Prove $\neg(xy > 100) \equiv xy \leq 100$.

Proof: Take two positive real numbers x, y.

Assume that $\underline{\underline{x \leq 10}}$ and $y \leq 10$.

$x \cdot y \leq 10 \cdot y$ because $y > 0$

$x \cdot y \leq 10 \cdot y \underbrace{\leq 10 \cdot 10}_{\text{because } y \leq 10} = 100$

Therefore $xy \leq 100$.

Theorem For all integers n, if n^2 is odd then n is odd.

Proof Assume n is not odd will prove n^2 is not odd.

n is even. There is an integer c such $n = 2c$.

Then $n^2 = (2c)^2 = 2 \cdot 2c^2$

Since $2c^2$ is an integer n^2 is even and not odd

$\square$

If: For any integer a, if a^2 is even then 4 evenly divides a^2.

Proof: → We will show that if a^2 is even then a is even. and (P, Q)

→ if a is even then a^2 is evenly divided by 4. (Q, R)

✓ $P \to Q$ and $Q \to R$ $\Rightarrow$ $P \to R$.

by contra   direct.

Assume a is odd will prove a^2 is odd.

$a = 2c+1$ for some integer c.

$a^2 = (2c+1)^2 = 4c^2 + 4c + 1 = 2(2c^2 + 2c) + 1$

Since $2c^2 + 2c$ is an int, a^2 is odd.

If a is even then 4 evenly divides a^2.

$a = 2c$ for some c.

$a^2 = (2c)^2 = 4c^2$ $\quad$ c^2 is an integer

so 4 evenly divides a^2.

$\boxed{X}$