$$\begin{bmatrix} 0 & 2 & 1 & -8 \\ 1 & 2 & -3 & 0 \\ -1 & 1 & 2 & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -2 & -3 & 0 \\ 0 & 2 & 1 & -8 \\ 0 & 0 & -1/2 & -1 \end{bmatrix}$$

$$\begin{array}{cccc} \uparrow & \uparrow & \uparrow & \end{array}$$

pivot cols

$$\begin{bmatrix} 1 & -2 & -3 & 0 \\ 0 & 1 & 1/2 & -4 \\ 0 & 0 & 1 \;{\scriptstyle \cdot -1/2} & 2 \;{\scriptstyle \cdot -1} \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & -3 & 0 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 \;{\scriptstyle 3} & 2 \;{\scriptstyle 6} \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 0 & 6 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Reduced Row Ech Form.

$$\begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$x_1 = -4$
$x_2 = -5$
$x_3 = 2.$

$-12 - -6 = -6$

$$\begin{bmatrix} 1 & -2 & -6 & 12 \\ 1 & -4 & -12 & 22 \\ 2 & 4 & 12 & 17 \end{bmatrix} \begin{matrix} - \\ -2 \end{matrix}$$

$$\begin{bmatrix} 1 & -2 & -6 & 12 \\ 0 & -2 & -6 & 10 \\ 0 & 8 & 24 & -7 \end{bmatrix} +4$$

Inconsistent system

$$\begin{bmatrix} 1 & -2 & -6 & 12 \\ 0 & -2 & -6 & 10 \\ 0 & 0 & 0 & 33 \end{bmatrix} \rightarrow 0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 33.$$

If the right-most col of an augmented matrix is a pivot ~~col~~ column, the system is inconsistent.

$$\begin{bmatrix} 0 & 1 & 5 & -6 \\ 1 & 4 & 3 & -2 \\ 2 & 7 & 1 & 2 \end{bmatrix} \text{Swap.}$$

$$\begin{bmatrix} 1 & 4 & 3 & -2 \\ 0 & 1 & 5 & -6 \\ 2 & 7 & 1 & 2 \end{bmatrix} -2$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 4 & 3 & -2 \\ 0 & 1 & 5 & -6 \\ 0 & -1 & -5 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 3 & -2 \\ 0 & 1 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad -4$$

$$\begin{bmatrix} 1 & 0 & -17 & 22 \\ 0 & 1 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 - 17x_3 = 22$$
$$x_2 + 5x_3 = -6.$$

pivot cols $\Rightarrow$ basic variables.
non-pivot cols $\Rightarrow$ free variables.

For any set of values for the free variables, there is a unique setting for the basic variables that satisfies the system.

Free variables $\Rightarrow$ Infinite # solutions.
+ consistent

Parametric description:
$$x_1 = 17x_3 + 22$$
$$x_2 = -5x_3 - 6.$$

Thm A linear system is consistent if and only if the right column of the augmented matrix is not a pivot col.

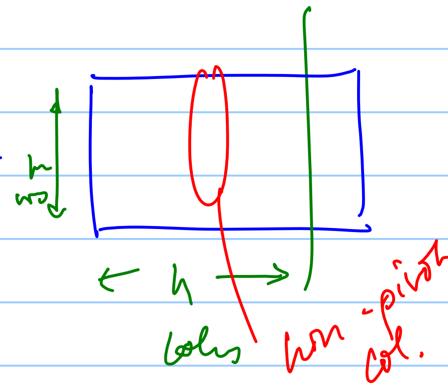
(echelon form does not have $0 \cdots 0 \; b \neq 0$)

If a linear system is consistent:

Unique soln if there are no free variables.
Infinite set of solns if there are free vars.

Question

Linear System   consistent
$m < n$



Vectors:

vector is matrix with one column

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} \qquad u_i \in \mathbb{R}$$

real numbers.

$$\vec{u} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Set of all Vectors w/ two entries, $= \mathbb{R}^2$

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Vector addition. $\vec{u}, \vec{v} \in \mathbb{R}^2$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \end{bmatrix}.$$

Scalar multiplication   Scalar : number (not a vector).

$\vec{u} \in \mathbb{R}^2$ $c \in \mathbb{R}$ $c \cdot \vec{u}$

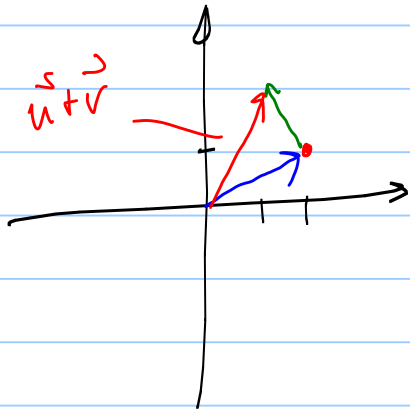
$$\vec{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \qquad 5 \cdot \vec{u} = \begin{bmatrix} 15 \\ 10 \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$3 \cdot \vec{u} + 4 \cdot \vec{v} =$$

$$3 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + 4 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$$

in $\mathbb{R}^2$ $\qquad \vec{u} \in \mathbb{R}^2$ point in plane.

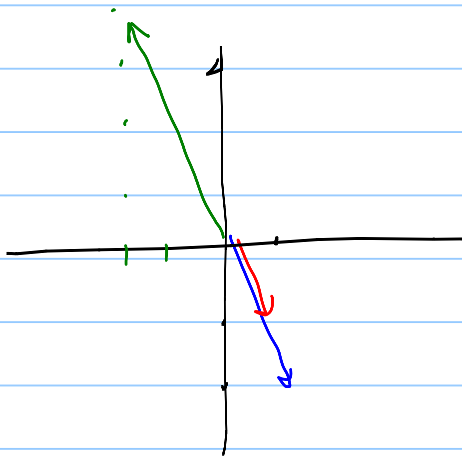


$\vec{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\vec{u}+\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\vec{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$



$\frac{1}{2}\vec{u} = \begin{bmatrix} 1/2 \\ -1 \end{bmatrix}$

$-2\vec{u} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$

$\mathbb{R}^3$ points 3-space.

$\mathbb{R}^n$ $\quad n>0$ $\quad$ is the set of all vectors of ordered n-tuples

$[u_1\ u_2\ \dots\ u_n] \neq (u_1, u_2, \dots, u_n) = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$

$\vec{0} = (0, 0, \dots, 0).$

Algebraic Properties of $\mathbb{R}^n$: $\quad \vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$
$c, d \in \mathbb{R}$.

Comm: $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
Assoc: $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$.

$$\vec{u} + 0 = \vec{u}$$
$$\vec{u} - \vec{u} = \vec{u} + (-1)\vec{u} = \vec{0}$$

$$c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}.$$
$$(c+d)\vec{u} = c\vec{u} + d\vec{u}$$
$$c(d\vec{u}) = (cd)\vec{u}$$

$$1 \cdot \vec{u} = \vec{u}.$$

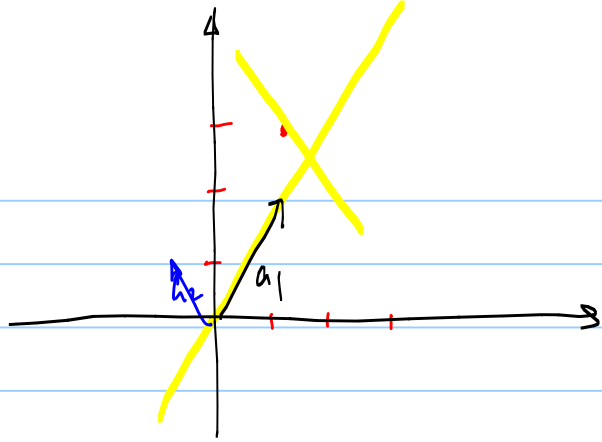
$V_1, V_2, \ldots, V_p \in \mathbb{R}^n \qquad c_1, \ldots, c_p \in \mathbb{R}$.

$$\vec{y} = c_1 \vec{V_1} + c_2 \vec{V_2} + \cdots + c_n \vec{V_p}$$

$\vec{y}$ is a linear combination of $\vec{V_1} \cdots \vec{V_p}$
with weights $c_1, \ldots, c_p$

$\vec{a}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \vec{a}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Q: Is there a linear combination of $\vec{a}_1$ and $\vec{a}_2$ that is $= b$ $\quad b = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$?



Is there an x_1 & x_2 such that.

$$x_1 \vec{a}_1 + x_2 \vec{a}_2 = \vec{b} \ ?$$

$$x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$