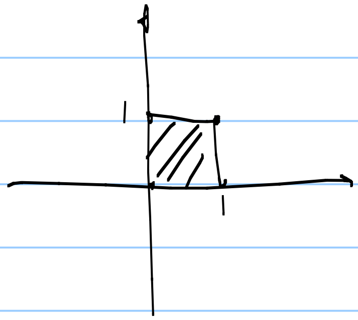
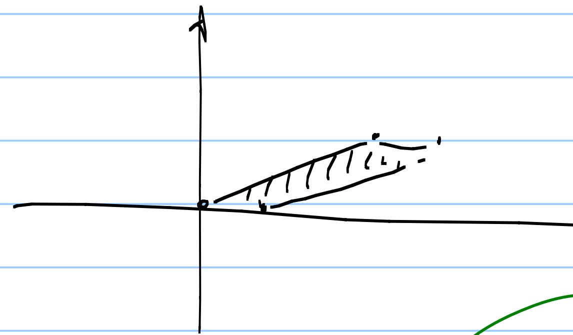
Note Title 10/11/2013

A: $m \times n$ matrix $\qquad T: \mathbb{R}^n \to \mathbb{R}^m$

$$T(\vec{x}) = A\vec{x}$$

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \qquad T: \mathbb{R}^2 \to \mathbb{R}^2$$





$\cancel{f(x) = x^2}$

$$A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v}$$
$$A(c\vec{v}) = cA\vec{v}.$$

T defined multiplication by A

$$\left.\begin{aligned} T(\vec{u} + \vec{v}) &= T(\vec{u}) + T(\vec{v}) \\ T(c\vec{v}) &= cT(\vec{v}) \end{aligned}\right\}$$

T is linear if it satisfies these two properties for all $c, \vec{u}, \vec{v}$.

$$T(\vec{0}) = T(0 \cdot \vec{v}) = 0 \cdot T(\vec{v}) = \vec{0}.$$

$$T(c\vec{u} + d\vec{v}) = T(c\vec{u}) + T(d\vec{u}) = cT(\vec{u}) + dT(\vec{v}).$$

$$T(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = c_1T(\vec{v}_1) + \cdots + c_nT(\vec{v}_n).$$

Will see: T linear $\longrightarrow$ defined by matrix mult.
$T: \mathbb{R}^n \to \mathbb{R}^m$

$$\begin{bmatrix}2\\4\\6\end{bmatrix} = 2 \cdot \begin{bmatrix}1\\2\\3\end{bmatrix}$$

Application

Widget A
Widget B.

For \$1 worth of A Company spends:

45¢	material	Widget B:	40¢
25¢	labor		35¢
15¢	overhead		15¢

a amount Widget A
b amount Widget B.

$$\begin{bmatrix}.45 & .40\\ .25 & .35\\ .15 & .15\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix} = \begin{bmatrix}\square\\ \square\\ \square\end{bmatrix}\begin{array}{l}\text{total amt of material}\\ \text{labor}\\ \text{overhead.}\end{array}$$

$T: \mathbb{R}^n \to \mathbb{R}^m$ linear.

T is completely determined by its action on the columns of an $n \times n$ identity matrix.

$$I_n = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix} \qquad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$T: \mathbb{R}^n \to \mathbb{R}^n$ defined by mult times I_n.
$T(\vec{x}) = \vec{x}$.

Column j of I_n $\vec{e}_j$ all 0's except for a 1 in location j.

$\vec{e}_3 \in \mathbb{R}^4$ $\quad \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ $\qquad \vec{e}_2 \in \mathbb{R}^5$ $\quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$T: \mathbb{R}^n \to \mathbb{R}^m$ $\qquad T(\vec{e}_1) \quad \dots \quad T(\vec{e}_n)$

Suppose $T: \mathbb{R}^2 \to \mathbb{R}^3$ $\qquad T(\vec{e}_1) = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$ $\quad T(\vec{e}_2) = \begin{bmatrix} 8 \\ 3 \\ 4 \end{bmatrix}$

$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$T(\vec{x})$ $\qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x_1 \vec{e}_1 + x_2 \vec{e}_2$

$$T(\vec{x}) = T(x_1 \vec{e}_1 + x_2 \vec{e}_2) = x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2)$$

$$= x_1 \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 8 \\ 3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 8 \\ -1 & 3 \\ 2 & 4 \end{bmatrix} \vec{x}$$

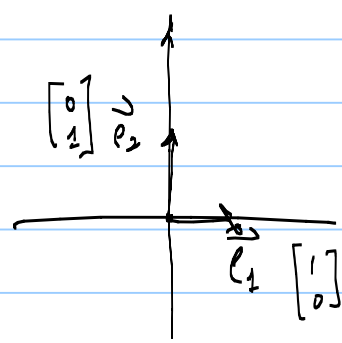
<u>Theorem</u> $\quad T: \mathbb{R}^n \to \mathbb{R}^m \qquad \vec{e}_1 \dots \vec{e}_n \in \mathbb{R}^n$

$$T(\vec{e}_j) \in \mathbb{R}^m$$

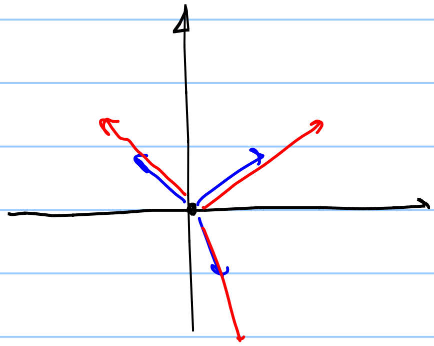
There exists a <u>unique</u> $m \times n$ matrix A s.t.

$$T(\vec{x}) = A\vec{x} \quad \forall \vec{x} \in \mathbb{R}^n$$

$$A = \begin{bmatrix} T(\vec{e}_1) & \dots & T(\vec{e}_n) \end{bmatrix}$$



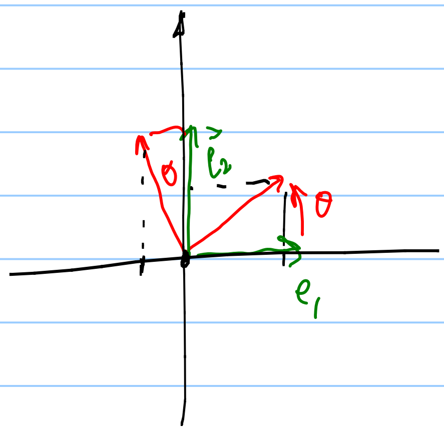
$$T(\vec{x}) = 2\vec{x} \qquad T(\vec{0}) = \vec{0}$$



$$T(\vec{e}_1) = 2\vec{e}_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$T(\vec{e}_2) = 2\vec{e}_2 = 2\cdot\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

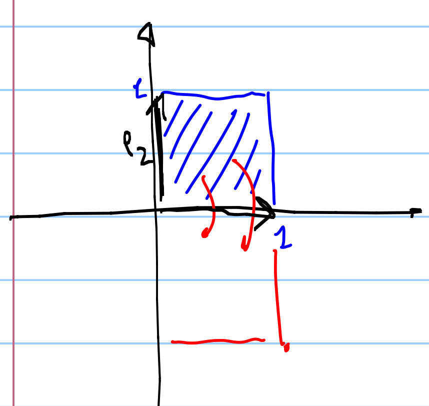
$$A : \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \qquad T(\vec{x}) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\vec{x}.$$



$$T(\vec{e}_1) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

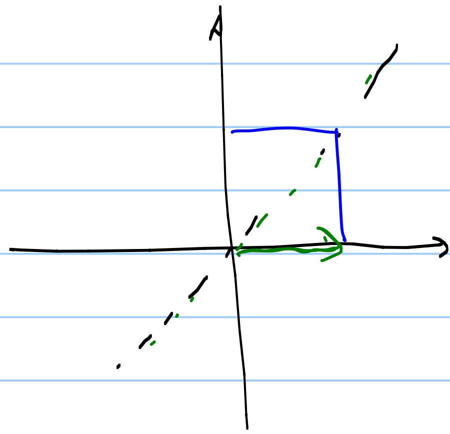
$$T(e_2) = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$



$$T(\vec{e}_1) = \vec{e}_1 \qquad A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$T(\vec{e}_2) = -\vec{e}_2$$

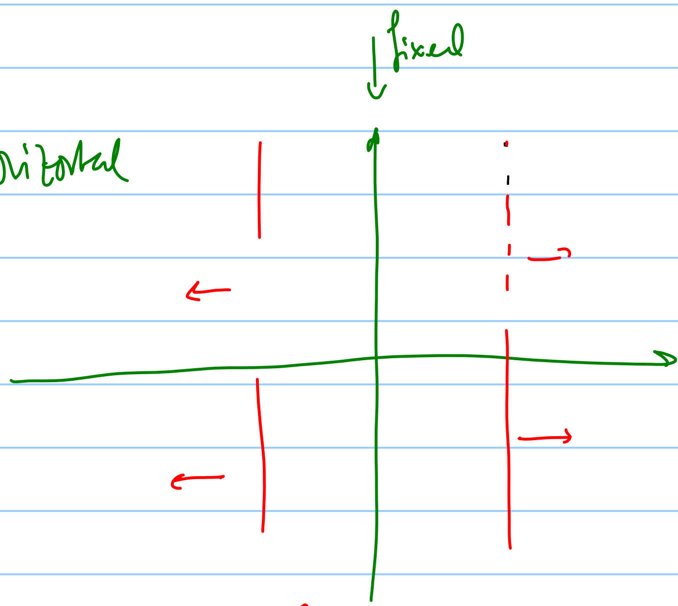


$$T(\vec{e}_1) = \vec{e}_2$$

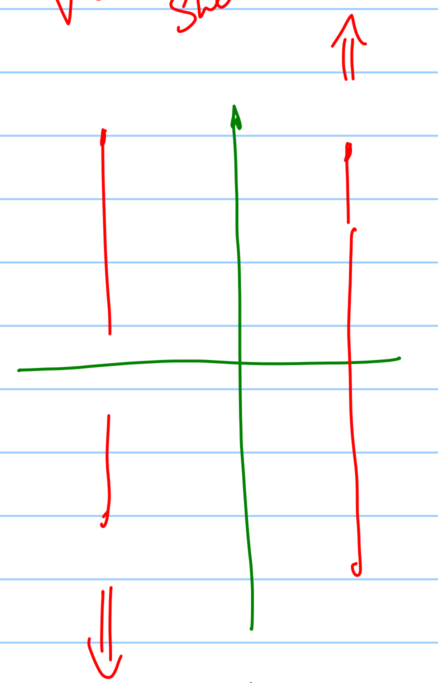
$$T(\vec{e}_2) = \vec{e}_1$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

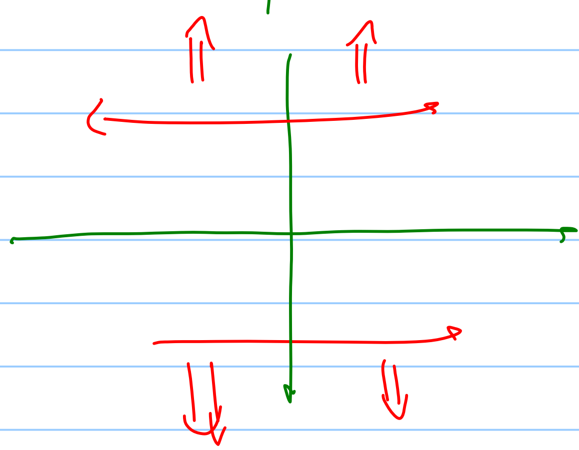
Stretch Horizontal

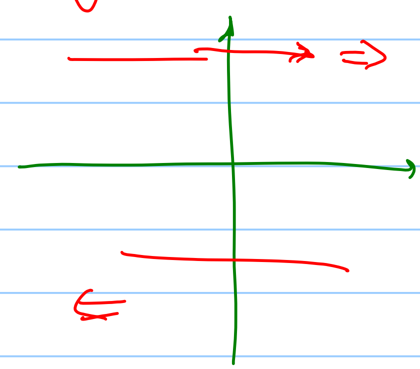


Vert shear.

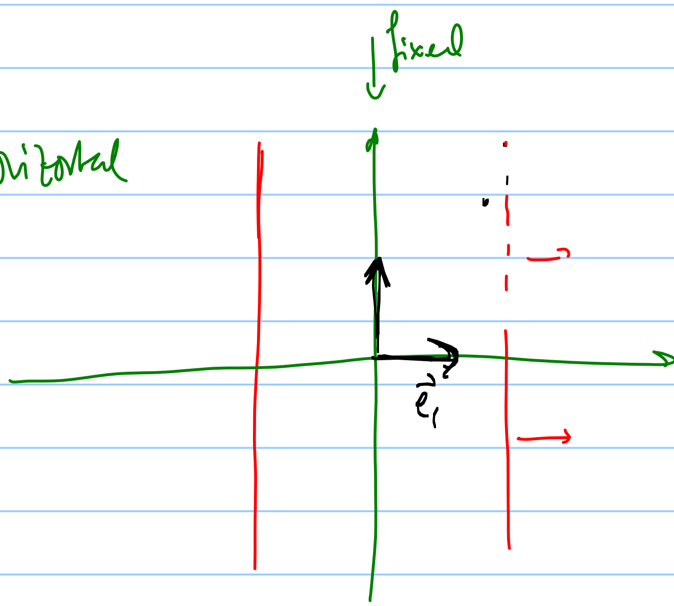


Vert Stretch





Stretch Horizontal



$T(\vec{e}_1) = 3\vec{e}_1$

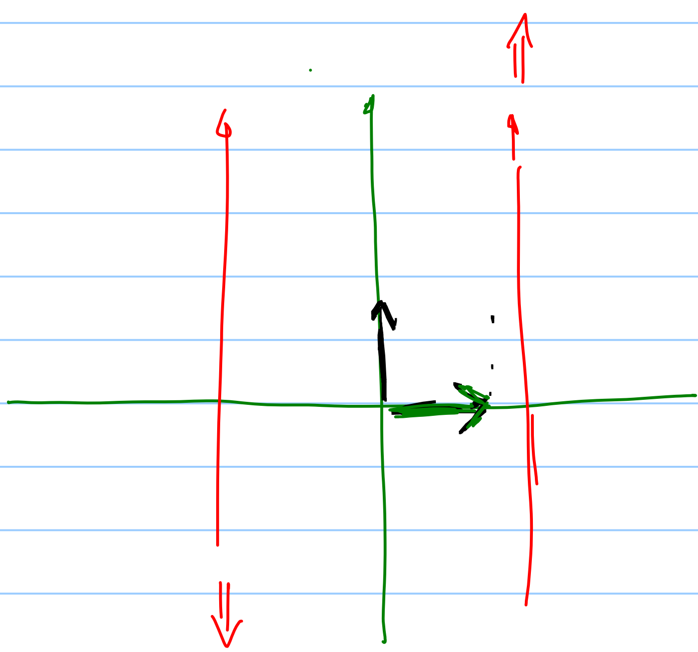
$T(\vec{e}_2) = e_2$

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$$

$k > 1$ stretch

$k < 1$ squish shrink.



$$T(e_1) = \begin{bmatrix} 1 \\ k \end{bmatrix}$$

$$T(e_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$$