Section 2.2, cont. Invertible Matrices.

Note Title 10/21/2013

A is a $n \times n$ matrix is invertible if there is an A^{-1}

$$A^{-1} \cdot A = A \cdot A^{-1} = I_n.$$

A is invertible iff the eq $A\vec{x} = \vec{b}$ has a unique solution for every $\vec{b}$.

$\Rightarrow$ If A is invertible then A^{-1} is also invertible

$$(A^{-1})^{-1} = A$$

$\Rightarrow$ $(AB)^{-1}$ A & B invertible.

$$B^{-1}A^{-1}(AB) = I_n.$$

$$B^{-1} I B$$

$$B^{-1} B$$

$$I_n.$$

$$(AB)^{-1} = B^{-1}A^{-1}$$

$\Rightarrow$ A is invertible then so is A^T

$$(A^T)^{-1} = (A^{-1})^T$$

$$\underbrace{C^{-1}B^{-1}A^{-1}}(ABC) = I_n$$

If $A_1 \cdots A_n$ all invertible

$$(A_1 \cdots A_n)^{-1} = A_n^{-1} A_{n-1}^{-1} \cdots A_2^{-1} A_1^{-1}$$

A $n \times n$ matrix is invertible iff the red. row ech form of A is I_n

$\equiv$ A is row equivalent to I_n

Row Ops   Row replacement
Scalar mult
Swap.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[R_3 \leftarrow R_3 + 3 \cdot R_1]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

A is 3x3 matrix

$$[A] \xrightarrow{\text{row op}} [A']$$

$$E \cdot A \qquad I_3 \xrightarrow{\text{row op}} E,$$

Elem matrix: result of one row op performed on I_n.

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \longrightarrow \begin{bmatrix} a & b & c \\ d & e & f \\ g+3a & h+3b & i+3c \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g+3a & h+3b & i+3c \end{bmatrix}$$

Scalar mult

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \rightsquigarrow \begin{bmatrix} a & b & c \\ -2d & -2e & -2f \\ g & h & i \end{bmatrix}$$

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \right) /\!/$$

Elem matrix for swap

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A is $n \times n$ matrix and it's invertible.

$$\begin{bmatrix} A \end{bmatrix} \xrightarrow{R_1 R_2 \cdots R_p} \begin{bmatrix} I \end{bmatrix}.$$

Let E_i be the elem matrix for row op R_i

$$\underbrace{\left(E_p \cdot \quad \cdot E_2 \cdot E_1\right)}_{A^{-1}} A = I_n.$$

$$n \underbrace{\begin{bmatrix} A & I \end{bmatrix}}_{\longleftarrow 2n \longrightarrow} \leadsto \begin{bmatrix} I & A^{-1} \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 2 & 1 & 0 & 0 \\ 1 & -2 & -3 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & 1 & 1 & 0 \\ 0 & 0 & +1 & -2 & -2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & 1 & 1 & 0. \\ 0 & 2 & 1 & 0 & 0 & 1. \end{bmatrix}, \quad \begin{bmatrix} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & -1 & -1 \\ 0 & 0 & 1 & -2 & -2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & -1 & -1 \\ 0 & 0 & 1 & -2 & -2 & -1 \end{bmatrix} \qquad \begin{bmatrix} -1 & 0 & 0 & 4 & 3 & 1 \\ 0 & -1 & 0 & -1 & -1 & -1 \\ 0 & 0 & 1 & -2 & -2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 0 & 5 & 4 & 2 \\ 0 & -1 & 0 & -1 & -1 & -1 \\ 0 & 0 & 1 & -2 & -2 & -1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & -4 & -3 & -1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -2 & -2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 2 \\ 1 & -2 & -3 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} -4 & -3 & -1 \\ 1 & 1 & 1 \\ -2 & -2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2.3. Characteristics of Inv. Matrices.

Let A be an $n \times n$ matrix then the following are all equiv (all true or all false).

a) A is invertible.
b) A is row equiv to I_n
c) A has n pivots (n pivot rows + n pivot cols)
d) The equation $A\vec{x} = \vec{0}$ has only trivial Solution. ←
e) The columns of A are linearly independent.
f) $T(\vec{x}) = A\vec{x}$ T is one-to-one.
g) $A\vec{x} = \vec{b}$ Consistent for every $\vec{b}$.
h) Col of A span $\mathbb{R}^n$.
i) $T(\vec{x}) = A\vec{x}$ is onto.
j) There is an $n \times n$ matrix C s.t. $CA = I$.
k) there is an $n \times n$ matrix D s.t. $AD = I$. } $C = D$.
l) A^T is invertible.

$T: \mathbb{R}^n \to \mathbb{R}^n$ if T is linear. then

$$T(\vec{x}) = \underline{A}\vec{x}.$$

T is invertible if there is an $S: \mathbb{R}^n \to \mathbb{R}^n$

$$\vec{x} = T(S(\vec{x})) = S(T(\vec{x}))$$

$$S(\vec{y}) = A^{-1}\vec{x}.$$

$$\underbrace{E_p \cdots E_2 E_1}_{A^{-1}} A = I_n.$$

$E_p \cdots E_1 =$ Row ops applied to I_n.