5.1 Eigenvalues & Eigenvectors

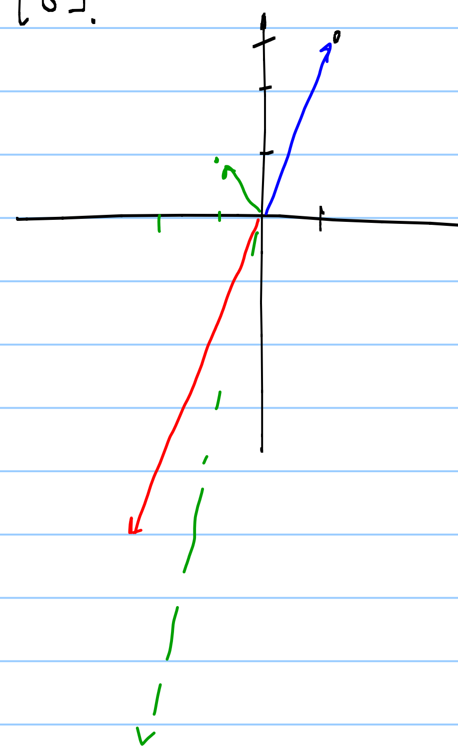
Note Title 11/13/2013

$A = \begin{bmatrix}1 & -1\\ 6 & -4\end{bmatrix}$ $\qquad T: \mathbb{R}^2 \to \mathbb{R}^2$

$A\begin{bmatrix}1\\3\end{bmatrix} = -2\begin{bmatrix}1\\3\end{bmatrix}$

$$\begin{bmatrix}1 & -1\\ 6 & -4\end{bmatrix}\begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}-2\\-6\end{bmatrix} = -2\begin{bmatrix}1\\3\end{bmatrix}.$$

$$\begin{bmatrix}1 & -1\\ 6 & -4\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}-2\\-10\end{bmatrix}$$



$$\underline{A}\,\underline{\vec{v}} = \underline{\lambda}\,\underline{\vec{v}}$$
scalar.

$$A(c\vec{v}) = \lambda c\vec{v}$$

An eigenvector of an $n \times n$ matrix A, is a non-zero vector $\vec{x}$ such that:

$$A\vec{x} = \lambda\vec{x} \quad \text{for some scalar } \lambda.$$

λ is called an eigenvalue of A

$\vec{x}$ is the eigenvector corresponding to eigenvalue λ.

Easy to test if $\vec{x}$ is an eigenvector of A.

$$A\vec{x} =$$

eigenvector??

$$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4 \begin{bmatrix} 6 \\ -5 \end{bmatrix}$$

Show that 7 is an eigenvalue of $\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$

$\exists \vec{x}$

$$\underbrace{\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}}_{A} \vec{x} = 7\vec{x}$$

$$A\vec{x} = 7\vec{x}$$

$$A\vec{x} - 7\vec{x} = 0$$

$$A\vec{x} - 7I\vec{x} = 0$$

$$I \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$(A - 7I)\vec{x} = 0$$

$$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix}$$

Find $\vec{x}$ s.t. $\begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix} \vec{x} = \vec{0}$

$$\begin{bmatrix} -6 & 6 & 0 \\ 5 & -5 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{matrix} x_1 - x_2 = 0 \\ x_1 = x_2 \end{matrix}$$

$$(x_2)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Verify $\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \end{bmatrix} = 7\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Can verify $\vec{x}$ eigenvector of A

Given eigenvalue λ of A $\rightarrow$ verify λ is an eigenvalue

find corresponding eigenvector

A , λ

$A\vec{x} = \lambda\vec{x}$ has a solution?

$\Updownarrow$

$\underline{(A - \lambda I)}\,\vec{x} = \vec{0}$ has a non-trivial solution?

$\Updownarrow$

$(A - \lambda I)$ has at least one free variable

$\Updownarrow$

$A - \lambda I$ not invertible

$\Updownarrow$

$\det(A - \lambda I) = 0$.

$$A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$$

2 is an eigenvalue.

$$\underbrace{A - 2I} \quad \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Free: x_2 & x_3

$$x_1 = x_2 - 6x_3$$

$$\begin{bmatrix} x_2 - 6x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -6 \\ 0 \\ 1 \end{bmatrix}$$

$$(A - 2I)\vec{x} = \vec{0}$$

$$\underbrace{\text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ 0 \\ 1 \end{bmatrix} \right\}}_{\text{Eigenspace corresponding to eigenvalue 2.}}$$

are all eigenvectors corresponding to eigenvalue 2.

Eigenspace corresponding to eigenvalue 2.

Theorem 1: Eigenvalues of a triangular matrix are entries along the diagonal:

$$\begin{bmatrix} a_{11} - \lambda & * & * \\ 0 & a_{22} - \lambda & * \\ 0 & 0 & a_{33} - \lambda \end{bmatrix}$$

det is zero if a diag. entry is 0.

$\Updownarrow$

λ is an eigenvalue.

When is 0 an eigenvalue of A?

$$A\vec{x} = 0 \cdot \vec{x} = \vec{0} \qquad A\vec{x} = \vec{0}$$

non-trivial soln.

0 is an eigenvalue of A iff A is not invertible
iff $\det A = 0$.

Theorem: $\vec{v}_1 \cdots \vec{v}_r$ eigenvectors corresponding to distinct eigenvalues of an $n \times n$ matrix A.

$$\left.\begin{array}{l} \vec{v}_1 \longleftrightarrow \lambda_1 \\ \vec{v}_2 \longleftrightarrow \lambda_2 \\ \vdots \\ \vec{v}_r \longleftrightarrow \lambda_r \end{array}\right\} \text{all different.}$$

Then: $\vec{v}_1\ \vec{v}_2 \cdots \vec{v}_r$ are linearly indep.

Suppose $\vec{v}_1 \cdots \vec{v}_r$ are linearly dep.

Pick smallest p such that $\vec{v}_p = c_1\vec{v}_1 + c_2\vec{v}_2 \cdots + c_{p-1}\vec{v}_{p-1}$

mult by A: $A\vec{v}_p = c_1 A\vec{v}_1 + \cdots + c_{p-1} A\vec{v}_{p-1}$

$$\lambda_p\vec{v}_p = c_1\lambda_1\vec{v}_1 + \cdots + c_{p-1}\lambda_{p-1}\vec{v}_{p-1}$$

$$-\quad \lambda_p\vec{v}_p = c_1\lambda_p\vec{v}_1 + \quad + c_{p-1}\lambda_p\vec{v}_{p-1}$$

$$\vec{0} = \underline{c_1(\lambda_1-\lambda_p)}\vec{v}_1 + \cdots \underline{c_{p-1}(\lambda_{p-1}-\lambda_p)}\vec{v}_{p-1}$$

$\hookrightarrow$ λ_i distinct, means non-trivial lin comb.

(5.2) Find eigenvalue:

$$A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$$

Find all scalars λ.

Such that:

$$\begin{bmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{bmatrix} \vec{x} = \vec{0}$$

has non-trivial solution.

$$\det \begin{bmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{bmatrix} = 0. \iff$$

$$\Updownarrow$$

$$(2-\lambda)(-6-\lambda) - (3\cdot 3) = 0$$

$$\lambda^2 + 4\lambda - 21 = 0$$

$$(\lambda + 7)(\lambda - 3) = 0$$

eigenvalues: $\lambda = -7$

$\lambda = 3$