5.2 Cont

Note Title 11/18/2013

Props of Dets

A & B are $n \times n$ matrices.

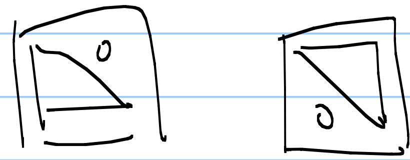
a) A is invertible iff $\det A \neq 0$.

b) $\det AB = \det A \det B$.

c) $\det A = \det A^T$

d) if A is triangular

$\det A =$ product of diagonal elts.



e) $A \rightsquigarrow B$ by one row op.

row rep: $\det A = \det B$

swap: $\det A = -\det B$

row scale by k: $\det B = k \cdot \det A$.

λ is an eigenvalue for $n \times n$ matrix A

iff $(A - \lambda I)\vec{x} = 0$ has a non-trivial solution.

iff $(A - \lambda I)$ not invertible

iff $\det(A - \lambda I) = 0$.

characteristic equation

polynomial in λ. n.

\# distinct real eigenvalues $\leq n$.

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 2 & 3 \\ 3 & -1 & 1 \end{bmatrix} \qquad A - \lambda I = \begin{bmatrix} 3-\lambda & 1 & 1 \\ 2 & 2-\lambda & 3 \\ 3 & -1 & 1-\lambda \end{bmatrix}$$

det

$$= (3-\lambda)\left[(2-\lambda)(1-\lambda)+3\right]$$
$$- \; 2\left[(1-\lambda) - -1\right]$$
$$+ \; 3\left[3-(2-\lambda)\right]$$

Let A & B be two $n \times n$ matrices.

A is "similar" to B if there is an invertible matrix P such that.

$$B = P^{-1}AP \qquad\qquad Q = P^{-1}$$
$$A = Q^{-1}BQ$$

$\Rightarrow$ If A & B are similar, they have same eigenvalues.

Show

$$B = P^{-1}AP \qquad \det(B - \lambda I) = \det(A - \lambda I)$$

If A & B similar, so are $A - \lambda I$ & $B - \lambda I$.

$$B - \lambda I = P^{-1}AP - \lambda P^{-1}P$$
$$= P^{-1}(AP - \lambda P)$$
$$= P^{-1}(A - \lambda I)P$$

$$\det(B - \lambda I) = \det(P^{-1})\det(A - \lambda I)\det(P)$$
$$= \det(A - \lambda I)\underbrace{\det(P^{-1})\det(P)}_{\substack{\det(P^{-1}P) \\ \det(I) \\ 1}}.$$

(5.3)

Suppose $A = P^{-1}DP$ similar to D
and D is diagonal matrix.

$$\begin{bmatrix} D_{} & & 0 \\ & \ddots & \\ 0 & & \square \end{bmatrix}$$

$$A^k = (P^{-1}DP)^k$$

$$A^3 = P^{-1}D\cancel{P} \cdot \cancel{P^{-1}}D\cancel{P} \cdot \cancel{P^{-1}}DP.$$
$$= P^{-1}D^3P$$

$$A^k = P^{-1}D^kP.$$

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad D \cdot D = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$D^k = \begin{bmatrix} 3^k & 0 & 0 \\ 0 & 2^k & 0 \\ 0 & 0 & 1^k \end{bmatrix}$$

A square matrix is diagonalizable if it is similar to a diagonal matrix D.

$$A = P^{-1} D P.$$

Theorem: A matrix A is diagonalizable iff it has n linearly indep eigenvectors.

$A = PDP^{-1}$ iff cols of P are linearly indep eigenvectors

D eigenvalues along the diagonal.

$$\begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{bmatrix}$$

$$A = PDP^{-1} \qquad P = [\vec{v}_1 \; \dots \; \vec{v}_n]$$

$$AP = PD$$

$$A [\vec{v}_1 \; \vec{v}_2 \cdots \vec{v}_n] = [\vec{v}_1 \; \vec{v}_2 \cdots \vec{v}_n] D.$$

$$\boxed{A\vec{v}_i = \lambda_i \vec{v}_i}$$

$$[A\vec{v}_1 \;\; A\vec{v}_2 \;\; A\vec{v}_n] = [\lambda_1\vec{v}_1 \;\; \lambda_2\vec{v}_2 \;\; \lambda_n\vec{v}_n]$$

To diagonalize A: Find eigenvalues. → diagonal of D

Find corresponding eigenvectors → columns of P.

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{bmatrix}$$

$$\underline{-(\lambda-1)(\lambda+2)^2} = 0$$

eigenvalues $\lambda = 1, -2.$

(2) Find eigenvectors.

$\lambda = 1$ $\quad A\vec{x} = \vec{x}$ $\quad$ non-trivial.

$$\underline{(A - I)}\vec{x} = \vec{0}$$

$$\begin{bmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{bmatrix}\vec{x} = \vec{0}.$$

$$\vec{x} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$ eigen vector.

$$\begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} \qquad \begin{bmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{bmatrix} \vec{x} = \vec{0}.$$

$$x_1 = -x_2 - x_3 \qquad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \vec{x} = 0.$$

$$\begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} +1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$A = PDP^{-1}$$