6.4

Note Title 11/27/2013

Start w/ basis for W subspace of $\mathbb{R}^4$

↳ Orthogonal Basis ⇒ Gram-Schmidt procedure.

Basis = $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_p\}$

Orthogonal = $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$

$$\vec{x}_1 \quad \vec{x}_2 \quad \ldots \quad \vec{x}_p$$

$$\vec{v}_1$$

Determine $\vec{v}_1 \ldots \vec{v}_i$ $\quad \text{Span}\{\vec{v}_1 \ldots \vec{v}_i\} = \text{Span}\{\vec{x}_1, \ldots, \vec{x}_i\} = W_i$

To get $\vec{v}_{i+1} = \vec{x}_{i+1} - \text{Proj}_{W_i} \vec{x}_{i+1}$

$$\vec{x}_{i+1} = \vec{z} + \text{Proj}_{W_i} \vec{x}_{i+1}$$

$\vec{z}$: orthogonal to W_i

Gram-Schmidt

$$\vec{v}_1 := \vec{x}_1$$

$$\vec{v}_2 := \vec{x}_2 - \left(\frac{x_2 \cdot v_1}{v_1 \cdot v_1}\right) \cdot v_1$$

$$\vec{v}_3 := \vec{x}_3 - \left(\frac{x_3 \cdot v_1}{v_1 \cdot v_1}\right) v_1 - \left(\frac{x_3 \cdot v_2}{v_2 \cdot v_2}\right) v_2$$

$$\vdots$$

$$\vec{V}_p = \vec{X}_p - \left(\frac{X_p \cdot V_1}{V_1 \cdot V}\right)V_1 - \left(\frac{X_p \cdot V_2}{V_2 \cdot V_2}\right)V_2 \cdots - \left(\frac{X_p \cdot V_{p-1}}{V_{p-1} \cdot V_{p-1}}\right)V_{p-1}$$

$$\vec{X}_1 = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} \qquad \vec{X}_2 = \begin{bmatrix} 5 \\ 6 \\ -7 \end{bmatrix}$$

$$\vec{V}_1 = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} \qquad \vec{V}_2 = \vec{X}_2 - \left(\frac{X_2 \cdot V_1}{V \cdot V_1}\right)V_1$$

$$= \begin{bmatrix} 5 \\ 6 \\ -7 \end{bmatrix} - \left(\frac{10}{\cancel{20}}\right)^{\frac{1}{2}} \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ -8 \end{bmatrix}$$

$$\vec{X}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} \qquad \vec{X}_2 = \begin{bmatrix} -5 \\ 1 \\ 5 \\ -7 \end{bmatrix} \qquad \vec{X}_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 8 \end{bmatrix}$$

$$\vec{V}_1 = \vec{X}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} \qquad \vec{V}_2 = \vec{X}_2 - \left(\frac{X_2 \cdot V_1}{V_1 \cdot V_1}\right)\vec{V}_1 = \begin{bmatrix} -5 \\ 1 \\ 5 \\ -7 \end{bmatrix} - \underbrace{\frac{-40}{20}}_{+2} \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -5+6 \\ 1+2 \\ 5-2 \\ -7+6 \end{bmatrix} = \begin{bmatrix} +1 \\ 3 \\ +3 \\ -1 \end{bmatrix}$$

$$v_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}$$

$$\vec{v_3} = \bar{X_3} - \left(\frac{X_3 \cdot V_1}{V \cdot V_1}\right) V_1 - \left(\frac{\bar{X_3} \cdot V_2}{V_2 \cdot V_2}\right) V_2$$

$$= \begin{bmatrix} 1 \\ 1 \\ -2 \\ 8 \end{bmatrix} - \frac{30}{20} \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} - \underbrace{\frac{-10}{20}}_{+\frac{1}{2}} \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 - 3\frac{3}{2} + \frac{1}{2} \\ 1 - \frac{3}{2} \cdot 1 + \frac{3}{2} \\ -2 - \frac{3}{2}(-1) + \frac{3}{2} \\ 8 - \frac{3}{2}(3) + \frac{1}{2}(-1) \end{bmatrix} =$$

$$\begin{bmatrix} -3 \\ 1 \\ 1 \\ 3 \end{bmatrix} = \vec{v_3}$$

$$\|v_3\| = \sqrt{(-3)^2 + 1^2 + 1^2 + 3^2}$$

$$= \sqrt{20}$$

$$\hookrightarrow \frac{1}{\sqrt{20}} \begin{bmatrix} -3 \\ 1 \\ 1 \\ 3 \end{bmatrix}$$

6.5 Least-Squared Approx.

$A\vec{x} = \vec{b}$ If inconsistent, find $\vec{x}$ such that $A\vec{x}$ is as close to $\vec{b}$ as possible.

Find $\vec{x}$ that minimizes: $\|A\vec{x} - \vec{b}\|$

A $m \times n$ matrix

$\vec{x} \in \mathbb{R}^n$ $\quad \vec{b} \in \mathbb{R}^m$. $\quad A\vec{x} \in$ Span of columns of A

The set of all possible $A\vec{x}$ = Col A.

The closest vector in subspace Col A to $\vec{b}$

is $\text{proj}_{\text{Col}A}\vec{b} = \vec{b}'$ ← In book the $\hat{b}$

Subspace W.
$\vec{y}$
closest vector in W to $\vec{y}$ is
$\text{proj}_W \vec{y}$.

Solutions to $A\vec{x} = \vec{b}'$

Set of solutions to this equation is the set of solutions to the least sq problem.

$\vec{b}$ $\vec{b}' = \text{proj}_{\text{Col}A}\vec{b}$

$\vec{b} = \vec{z} + \text{proj}_W \vec{b}$
$\vec{z}$ orthog to W.

$\underline{\vec{b} - \vec{b}'}$ is orthogonal to Col A.

$\vec{x}$ is a solution to $A\vec{x} = \vec{b}'$ $\iff$ $\vec{b} - \text{proj}_{\text{Col}A}\vec{b}$ is orthog to Col A

$\Updownarrow$

$\vec{x}$ a best approx to $A\vec{x} = \vec{b}$

$\iff \vec{a}_j \cdot (\vec{b} - \text{proj}_{\text{Col}A}\vec{b}) = 0 \quad \forall j.$

$\iff A^T(\vec{b} - \text{proj}_{\text{Col}A}\vec{b}) = 0.$

$$\begin{bmatrix} \vec{a}_1 \\ \vdots \\ \vec{a}_n \end{bmatrix} \begin{bmatrix} \vec{b} - \vec{b}' \end{bmatrix}$$

$$\Leftrightarrow \quad A^T(\vec{b} - \underline{A}\vec{x}) = 0$$

$$\Leftrightarrow \quad A^T\vec{b} - A^TA\vec{x} = 0$$

$$\Leftrightarrow \quad \underline{\underline{A^T\vec{b} = A^TA\vec{x}}}$$

Condition for $\vec{x}$ being a least sq solution.

$$A^TA\vec{x} = A^T\vec{b}$$

$$A = \begin{bmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 3 \\ 1 \\ -4 \\ 2 \end{bmatrix}$$

$$\underline{\underline{(A^TA)}}\,\vec{x} = \underline{\underline{(A^T\vec{b})}}$$

$$A^T$$

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ -2 & 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ -2 & 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -1 & 2 \\ 0 & 3 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 6 \\ 6 & 42 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 6 \\ 6 & 42 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \end{bmatrix}$$