ICS 6D **Due: Wednesday, October 22, 2014, 3:00PM**

Homework 2

Instructor: Sandy Irani

1. Give a recursive definition for strings of properly nested parentheses and curly braces. For example $(\{\})\{\}()$ is properly nested but $(\{)\}$ is not.

2. Let $B = \{0, 1\}$. In lecture, we discussed the recursive definition for B^*, the set of all binary strings of length 0 or larger. Recall that λ is the symbol for the empty string that has no characters. Here is the recursive definition for B^*:

   - λ is in B^*.
   - if $x \in B^*$, then $x0$ and $x1$ is also in B^*.

   Give a recursive definition for the set S which is all binary strings in which all the 0's come before all the 1's. For example $0011111 \in S$, but $0010111 \notin S$.

3. Suppose someone takes out a home improvement loan for \$30,000. The annual interest on the loan is 6% and is compounded monthly. The monthly payment is \$600. Let a_n denote the amount owed at the end of the n^{th} month. The payments start in the first month and are due the last day of every month.

   (a) Give a recurrence relation for a_n. Don't forget the base case.

   (b) Suppose that the borrower would like a lower monthly payment. How large does the monthly payment need to be to ensure that the amount owed decreases every month?

4. Give a recursive algorithm that takes as input two positive integers x and y and returns the product of x and y. The only arithmetic operations your algorithm can perform are addition or subtraction. Furthermore, your algorithm should have no loops.

5. Give a recursive algorithm that takes as input a non-negative integer n and returns a set containing all binary strings of length n. Here are the operations on strings and sets you can use:

   - Initialize an empty set S.
   - Add a string x to S.
   - $y := 0x$ (This operation adds a 0 to the beginning of string x and assigns the result to string y).
   - $y := 1x$ (This operation adds a 1 to the beginning of string x and assigns the result to string y).
   - return(S)

   Also, you can have a looping structure that performs an operation on every string in a set:

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   For every x in S
        //perform some steps with string x
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6. Give a recursive algorithm whose input is a, a real number and n a positive integer and whose output is a^{2^n}. **Note that the exponent of a is 2^n.**

For each of the inductive proofs below, clearly indicate the base case and inductive step. For the inductive step, state what you are assuming and what you are proving and show where you use the inductive hypothesis in your proof.

1. Prove that for any $n \geq 1$, 4 evenly divides $3^{2n} - 1$.
2. Prove that for any $n \geq 1$, 6 evenly divides $7^n - 1$.
3. Prove that for every positive integer n, $\sum_{j=1}^{n} j^3 = \left(\frac{n(n+1)}{2}\right)^2$.
4. Prove that for every positive integer n, $\sum_{j=1}^{n} j \cdot 2^j = (n-1) \cdot 2^{n+1} + 2$.
5. Prove that for $n \geq 2$, $n^3 \geq 2n + 4$.
6. Prove that for $n \geq 2$, $3^n > 2^n + n^2$.
7. Define the sequence $\{c_n\}$ as follows:
   - $c_0 = 5$
   - $c_k = (c_{k-1})^2$.

   Prove that for $n \geq 0$, $c_n = 5^{2^n}$.
8. Define the sequence $\{b_n\}$ as follows:
   - $b_0 = 1$
   - $b_k = 2 \cdot b_{k-1} + 1$.

   Prove that for $n \geq 0$, $b_n = 2^{n+1} - 1$.