Set Operations.

**Definition:** Let $A$ and $B$ be sets, the union of $A$ and $B$ denoted $A \cup B$ is the set:
$$\{x \mid x \in A \lor x \in B\}.$$  

**Ex:** $A = \{1, 2, 3, 5\}$, $B = \{3, 4, 5\}$, $A \cup B = \{1, 2, 3, 4, 5\}$

- **Diagram:**

**Definition:** Let $A$ and $B$ be sets, the intersection of $A$ and $B$ denoted $A \cap B$ is the set:
$$\{x \mid x \in A \land x \in B\}.$$  

**Ex:** $A = \{1, 2, 3, 5\}$, $B = \{3, 4, 5\}$, $A \cap B = \{3, 5\}$

- **Diagram:**

- **Diagram:**
Can combine multiple set operations to define a set:

\[ A \cup B \cap C. \]

Use parentheses to denote the order in which they are applied:

\[(A \cup B) \cap C\]
\[A \cup (B \cap C)\]
**Definition:** If $A$ is a set, then the complement of $A$ (with respect to $U$) denoted $\overline{A}$ is the set of elements in $U$ that are not in $A$:

$$\overline{A} = \{ x \in U \mid x \notin A \}$$

**Example:** Let $U$ be the set of positive integers less than 100.

$$A = \{ x \mid x > 70 \}$$
$$\overline{A} = \{ x \mid x \leq 70 \}$$
Set Difference:

Let $A$ and $B$ be sets.

The difference of $A$ and $B$, $A - B$, is the set of elements in $A$ that are not in $B$.

$$A - B = \{x \mid x \in A \land x \notin B\}$$

Note: $A - B = A \cap \overline{B}$. 
\[ U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \quad A \cup B = \{1, 2, 3, \ldots, 7, 8\} \]
\[ A = \{1, 2, 3, 4, 5\} \quad A \cap B = \{4, 5\} \]
\[ B = \{2, 3, 4, 7, 8\} \quad \overline{A} = \{0, 6, 7, 8, 9, 10\} \]
\[ \overline{A} \cup \overline{B} = \{0, 4, 5, 6, 7, 8, 9, 10\} \quad A - B = \{1, 2, 3\} \]
\[ A \cap \overline{B} = \{1, 2, 3\} \quad B - A = \{6, 7, 8\} \]
\[ \overline{B} = \{0, 1, 2, 3, 9, 10\} \quad \overline{A} \cup \overline{B} = \{0, 9, 10\} \]
\[(A \cup C) - B\]

\[(A \cap B) \cup (B \cap C)\]

\[(A - B) \cup C\]
Set Identities

We have defined set operations using logical operations. Can define set identities using the laws of logic.

Assume all sets are subsets of universe set $U$:

- $x \in A \cup B \iff x \in A \lor x \in B$
- $x \in A \cap B \iff x \in A \land x \in B$
- $\neg x \in A \iff x \not\in A$ or $\neg (x \in A)$

- $x \in \emptyset \iff F$
- $x \in U \iff T.$

Use identity law $p \land F = F$ to show $A \cap \emptyset = \emptyset$

- $x \in A \cap \emptyset \iff x \in A \land (x \in \emptyset)$
- $\iff x \in A \land F$
- $\iff F$
- $\iff x \not\in \emptyset$

$A \cap \emptyset = \emptyset.$

De Morgan's Law: $A \cup B = A \cap \overline{B}$

- $x \in A \cup B \iff x \not\in A \cup B \iff \neg (x \in A \lor x \in B)$
- $\iff \neg (x \in A) \land \neg (x \in B)$
- $\iff x \not\in A \land x \not\in B$
- $\iff x \in A \cap \overline{B}$

$A \cup B = A \cap \overline{B}.$
Cartesian Products.

An ordered pair of items: \((a, b)\)

\((a, b) \neq (b, a)\)

\[\mathcal{A}_1, \mathcal{A}_2 = \mathcal{A}_3, \mathcal{A}_3\]

Can also have ordered triplets:

\((a, b, c) \neq (c, b, a)\)

Ordered \(n\)-tuples: \((5, 1, 2, -3, \ldots, 17)\)

Ordered sets: \(\{a_1, a_2, \ldots, a_n\}\)

Cartesian Product:

Given the sets \(A \times B\), the Cartesian product of \(A\) and \(B\) (denoted \(A \times B\)) is the set of all pairs \((a, b)\) where \(a \in A\) and \(b \in B\).

\[\Rightarrow A \times B = \{ (a, b) \mid a \in A \text{ and } b \in B \}\]

Example: \(A = \{1, 2, 3\}, B = \{x, y, z\}\)

\[A \times B = \{ (1, x), (1, y), (1, z), (2, x), (2, y), (2, z) \}\]

\[A \Delta A \times B = \emptyset\]

\((1, y) \in A \times B\) \hspace{1cm} \( (y, 1) \notin A \times B\) \hspace{1cm} \(1 \notin A \times B\)
Cartesian Product (more examples).

\[ \mathbb{R} \times \mathbb{R} = \left\{ (x, y) \mid x \in \mathbb{R} \text{ and } y \in \mathbb{R}^3 \right\} \]

(all points in the plane.)

\[ \mathbb{Z} \times \mathbb{Z} = \left\{ (x, y) \mid x \in \mathbb{Z} \text{ and } y \in \mathbb{Z}^3 \right\} \]

Infinite 2-dimensional grid of points.

If \( A \neq B \), then \( A \times B \neq B \times A \)

Example: \( A = \{1, 2, 3\} \) \( B = \{x, y, z\} \)

\( (1, y) \in A \times B \) \( (y, 3) \in A \times B \) \( 1 \in A \times B \)

\( (y, 1) \in B \times A \) \( (1, y) \in B \times A \) \( 1 \in B \times A \) \( y \in B \times A \)

\( \text{Yes} \) \( \text{No} \) \( \text{No} \) \( \text{No} \)
$A \times A$ is sometimes denoted $A^2$.

Example: $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$.

For $n$ sets: $A_1, A_2, \ldots, A_n$

$A_1 \times A_2 \times \ldots \times A_n = \{ (a_1, a_2, \ldots, a_n) : a_i \in A_i \text{ and } a_2 \in A_2 \text{ and } \ldots \text{ and } a_n \in A_n \}$

Drinks = $\{ \text{ Milk, Soda, Coffee} \}$

Main = $\{ \text{ Hamburger, Chicken} \}$

Side = $\{ \text{ Fries, Fruit, Slaw} \}$

$\text{Main} \times \text{Side} \times \text{Drink} = \{ (\text{Hamburger, Fries, Milk}), (\text{Chicken, Slaw, Coffee}) \}$

$(\text{Chicken, Fruit, Soda}) \in \text{Main} \times \text{Side} \times \text{Drink}$

$A \times A \times A = A^3$

$A \times A \times A \times A = A^4$
Strings.

If $A$ is a set of symbols, elements in $A^n$ can be denoted by strings (without parens, commas.

Example: $A = \{a, b, 3\}$.

Elements in $A^3$ can be written as $\text{abb}$ instead of $(a, b, b)$.

What is $\{0, 1\}^3 = \{000, 001, 010, 011, 100, 101, 110, 111\}$

$D = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
$L = \{a, b, c, \ldots, z\}$.

Get $7 \in D \times L \times L \times L = D \times L^3$.

Strings of length 4 (digits or lower case letters):

$(DUL)^4$

$(DUL) \times (DUL) \times (DUL) \times (DUL)$
Length 2 or 3 starts w/ digit
(D x (DUL)) U (D x (DUL)^2).

= D x (DUL) U (DUL)^2.

Binary Strings

Length 5
30, 13^5

Length 5 or 6
30, 13^5 U 30, 13^6

Length 5 start w/ 0:
303 x 30, 13^4

Length 5, first & last bit are the same:
303 x 30, 13^3 x 303 U 313 x 30, 13^3 x 313

B = 30, 13

3 bit Strings B^3
Set $A$: Partition of $A$:
Set of Subsets $A_1$, ..., $A_n$

$A_i \cap A_j = \emptyset$ 
Each element $a \in A$ is an element of exactly one of the subsets.
$A_1 \cap A_2 \cdots \cap A_n = \emptyset$. 
$A_1 \cup \cdots \cup A_n = A$.

Example: $A =$ students enrolled in ICS 6B

$S_a =$ students whose last name starts with a $\in A$.

$S_z =$ students whose last names start with Z.

$S_a, S_b, S_c, \ldots, S_z$ form a partition of $A$.

Another example

$S_0 = \{ x \in \mathbb{N} : \text{remainder of } x \text{ divided by } 3 \text{ is } 0 \}$

$S_1 = \{ x \in \mathbb{N} : \text{remainder of } x \text{ divided by } 3 \text{ is } 1 \}$

$S_2 = \{ x \in \mathbb{N} : \text{remainder of } x \text{ divided by } 3 \text{ is } 2 \}$

$S_1, S_2, S_3$ form a partition of $\mathbb{N}$.
Functions

Continuous functions: 
\[ f(x) = x^2 - 3 \]
\[ f(x) = \sin(x) \]

Discrete math: functions whose inputs and outputs are from finite or countably infinite sets.

Example: map a set of computational tasks to computers in a distributed network.

The input/output relationship of a digital circuit (or computer program).

First part of definition: input & output sets.

\[ f: A \rightarrow B \]
\[ A \text{ is the domain} \]
\[ B \text{ is the target} \]
\[ A + B \text{ are sets.} \]

Example:
\[ f: \{0,1\}^5 \rightarrow \{0,1\}^4 \]
\[ f(01101) = 1001 \]

Input: binary strings length 5
Output: binary strings length 4.

\[ f \] must be well-defined for each element in the domain.
\[ \rightarrow \text{ one output in range for each input.} \]

Example:
\[ f: \mathbb{R} \rightarrow \mathbb{R} \]
\[ f(x) = \frac{1}{x} \]
\[ f(x) = \pm \sqrt{x^2 + 1} \]

Not defined for \( x = 0 \).
Two different outputs for each input.
Arrow diagram. \( X = \{ a, b, c, d \} \). \( g: X \rightarrow Y \)

\[
\begin{align*}
X & = \{ a, b, c, d \} \\
Y & = \{ 1, 2, 3, 4, 5 \}
\end{align*}
\]

Range of a function \( f \subseteq \text{Target} \). \( f: X \rightarrow Y \)

\[
\text{Range of } f = \{ y \in Y : \exists x \in X \text{ s.t. } f(x) = y \}
\]

For example above: \( \{ 1, 2, 3, 4, 5 \} \)

\( f: \mathbb{R} \rightarrow \mathbb{R} \)

\( f(x) = x^2 \)

range: non-negative real numbers.

\( g: \mathbb{Z} \rightarrow \mathbb{Z} \)

\( f(x) = 3x \)

range: multiples of 3.

\( f: \{ 0, 1 \}^3 \rightarrow \{ 0, 1 \}^3 \)

\( f(x) \): copy the first bit and append to the end.

\( f(00) = 000 \)

Range: \( \{ 000, 010, 101, 111 \} \)

\( f(01) = 010 \)

\( f(10) = 101 \)

\( f(11) = 111 \)

Range = Target: No.