Set Operations.

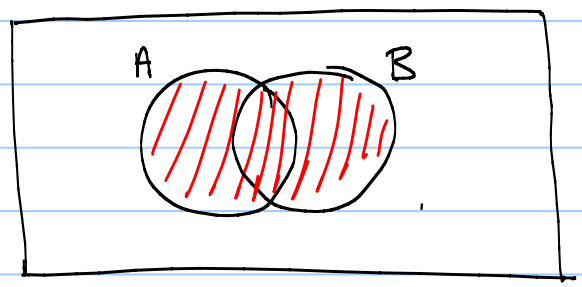
Note Title 10/19/2015

Definition: Let A & B be sets. The union of A and B denoted $A \cup B$ is the set:

$$\{x \mid x \in A \vee x \in B\}.$$

Ex: $A = \{1, 2, 3\}$.
$B = \{3, 4, 5\}$

$A \cup B = \{1, 2, 3, 4, 5\}$

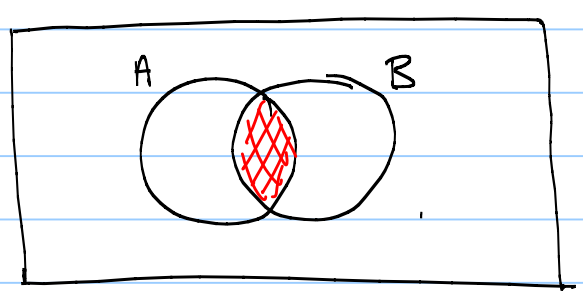


Definition: Let A & B be sets. The intersection of A and B denoted $A \cap B$ is the set:

$$\{x \mid x \in A \wedge x \in B\}.$$

Ex: $A = \{1, 2, 3\}$.
$B = \{3, 4, 5\}$

$A \cap B = \{3\}$

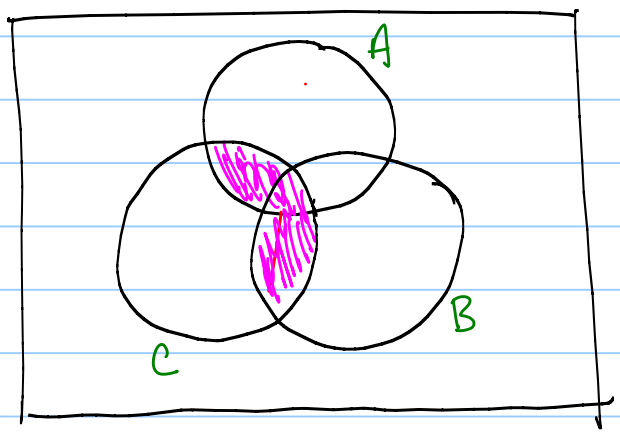


Can combine multiple set operations to define a set:

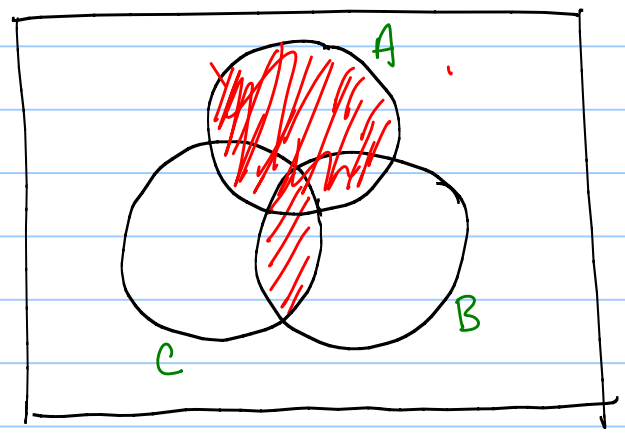
$$A \cup B \cap C.$$

Use parentheses to denote the order in which they are applied:

$(A \cup B) \cap C$



$A \cup (B \cap C)$



Definition: If A is a set, then the complement of A (with respect to U) denoted $\bar{A}$ is the set of elements in U that are not in A:

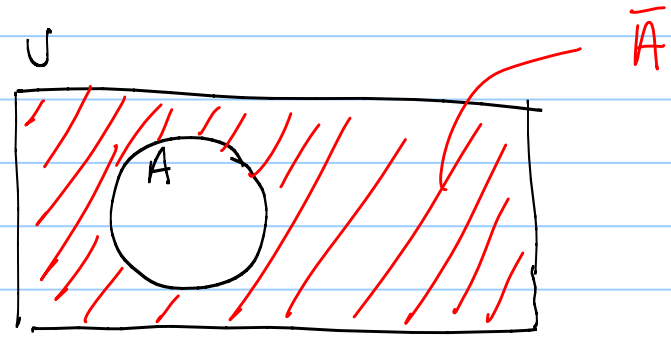
$\neg(x \in A)$.

$$\bar{A} = \{x \in U \mid x \notin A\}$$

Example: Let U be the set of positive integers less than 100.

$$A = \{x \mid x > 70\}$$

$$\bar{A} = \{x \mid x \leq 70\}$$

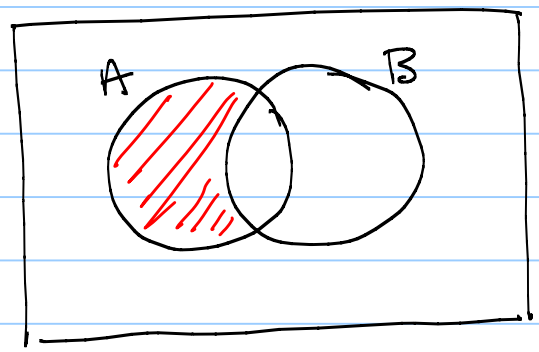


Set Difference:

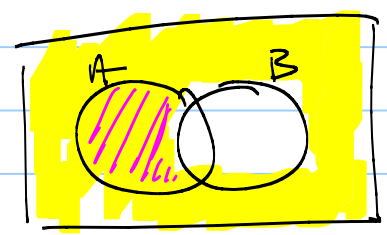
Let A and B be sets.

The difference of A and B, $A - B$, is the set of elements in A that are not in B.

$$A - B = \{x \mid x \in A \wedge x \notin B\}$$



Note: $A - B = A \cap \overline{B}$.



$U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$A = \{1, 2, 3, 4, 5\}$.

$B = \{4, 5, 6, 7, 8\}$

$\overline{A} = \{0, 6, 7, 8, 9, 10\}$

$\overline{A} \cup B = \{0, 4, 5, 6, 7, 8, 9, 10\}$

$A \cap \overline{B} = \{1, 2, 3\}$

$\overline{B} = \{0, 1, 2, 3, 9, 10\}$

$A \cup B = \{1, 2, 3, \ldots, 7, 8\}$

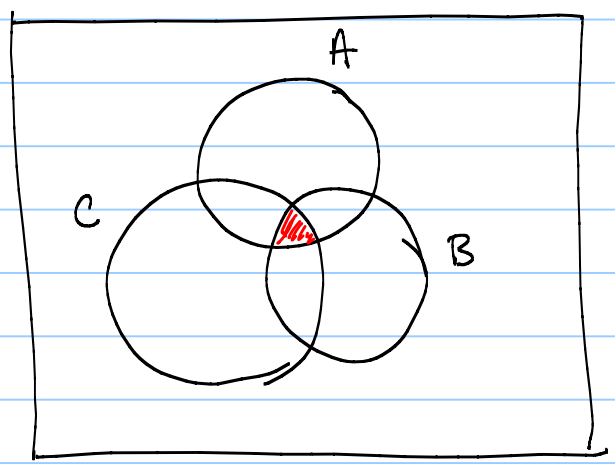
$A \cap B = \{4, 5\}$

$A - B = \{1, 2, 3\}$

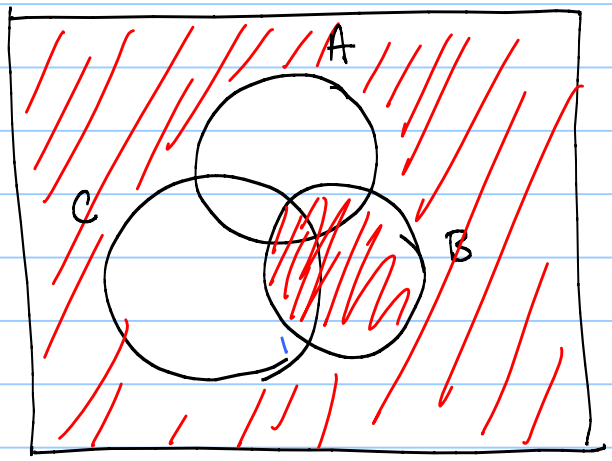
$B - A = \{6, 7, 8\}$

$\overline{A \cup B} = \{0, 9, 10\}$

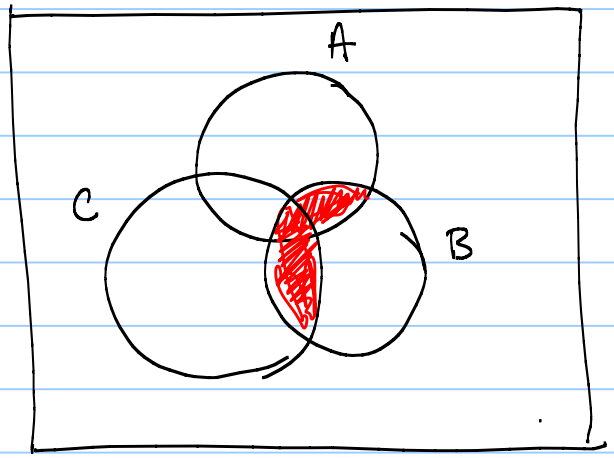
$A \cap B \cap C$



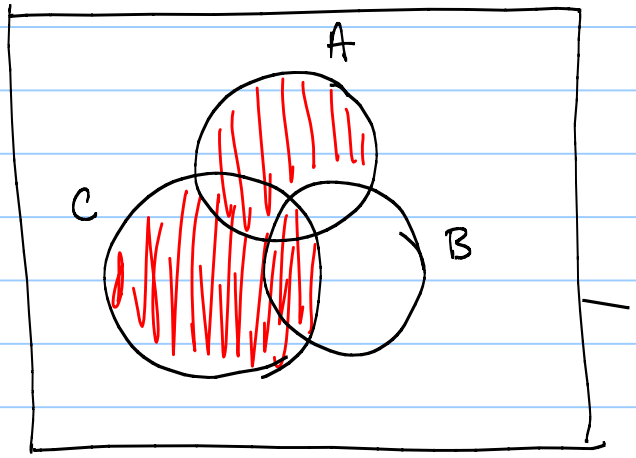
$\overline{(A \cup C) - B}$



$(A \cap B) \cup (B \cap C)$



$(A - B) \cup C$



Set Identities

We have defined set operations using logical operations. Can define set identities using the laws of logic.

Assume all sets are subsets of Universe set U:

$$x \in A \cup B \iff x \in A \vee x \in B$$
$$x \in A \cap B \iff x \in A \wedge x \in B$$
$$x \in \overline{A} \iff x \notin A \text{ OR } \neg(x \in A)$$

$$x \in \phi \iff F$$
$$x \in U \iff T.$$

Use identity law $p \wedge F \equiv F$ to show $A \cap \phi = \phi$

$$\begin{aligned} x \in A \cap \phi &\iff x \in A \wedge (x \in \phi) \\ &\iff x \in A \wedge F \\ &\iff F \\ &\iff x \in \phi \end{aligned}$$

$A \cap \phi = \phi.$

De Morgan's Law: $\overline{A \cup B} = \overline{A} \cap \overline{B}$

$$\begin{aligned} x \in \overline{A \cup B} &\iff x \notin A \cup B \iff \neg(x \in A \cup B) \\ &\iff \neg(x \in A \vee x \in B) \\ &\iff \neg(x \in A) \wedge \neg(x \in B) \\ &\iff x \in \overline{A} \wedge x \in \overline{B} \\ &\iff x \in \overline{A} \cap \overline{B} \end{aligned}$$

Cartesian Products.

Note Title 10/21/2015

An ordered pair of items: (a, b) ← regular parens, as opposed to curly braces denote the fact that order matters.

$$(a,b) \neq (b,a)$$

$$\{a,b\} = \{b,a\}$$

Can also have ordered tuples:

$$(a, b, c) \neq (c, b, a)$$

Ordered n-tuples: $(\underbrace{5, 1, 2, -3, \ldots, 17}_{n \text{ ordered items.}})$

Cartesian Product:

Given the sets A & B, the Cartesian product of A and B (denoted $A \times B$) is the set of all pairs (a,b) where $a \in A$ and $b \in B$.

$$\Rightarrow A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}.$$

Example: $A = \{1, 2\}$. $B = \{x, y, z\}$

$$A \times B = \{(1,x), (1,y), (1,z), (2,x), (2,y), (2,z)\}$$

$(1, y) \overset{?}{\in} A \times B$ yes

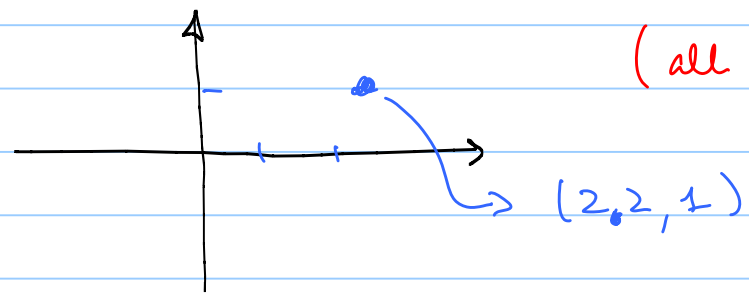
$(y, 1) \overset{?}{\in} A \times B$ no.

$1 \overset{?}{\in} A \times B$ no

$A \cap A \times B = \emptyset$

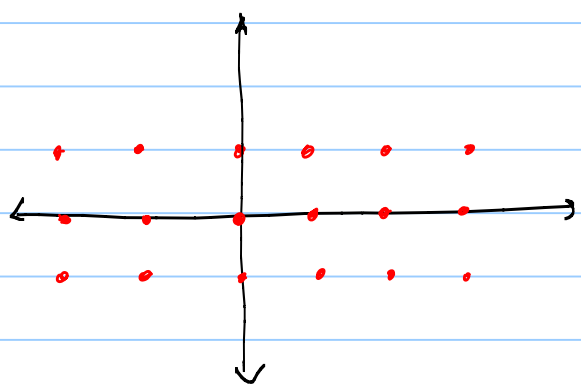
Cartesian Product (more examples).

$\mathbb{R} \times \mathbb{R} = \{ (x,y) \mid x \in \mathbb{R} \text{ and } y \in \mathbb{R} \}$



(all points in the plane.)

$\mathbb{Z} \times \mathbb{Z} = \{ (x,y) \mid x \in \mathbb{Z} \text{ and } y \in \mathbb{Z} \}$



Infinite 2-dimensional grid of points.

If $A \neq B$, then $A \times B \neq B \times A$

Example: $A = \{1, 2\}$. $B = \{x, y, z\}$

yes $(1, y) \overset{?}{\in} A \times B$ no $(y, 1) \overset{?}{\in} A \times B$ $1 \overset{?}{\notin} A \times B$

$(y, 1) \overset{?}{\in} B \times A$ $(1, y) \overset{?}{\in} B \times A$ $1 \overset{?}{\in} B \times A$ $y \overset{?}{\in} B \times A$.

yes no no no

$A \times A$ is sometimes denoted A^2.

Example: $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$.

For n sets: $A_1, A_2, \ldots, A_n$

$$A_1 \times A_2 \times \cdots \times A_n = \{ \overbrace{(a_1, a_2, \ldots, a_n)}^{\text{n-tuple.}} : a_1 \in A_1 \text{ and } a_2 \in A_2 \text{ ---- and } a_n \in A_n \}$$

Drink = {Milk, Soda, Coffee}

Main = {Hamburger, Chicken}

Side = {Fries, Fruit, Slaw}

$$\text{Main} \times \text{Side} \times \text{Drink} = \{ (\text{Hamburger}, \text{Fries}, \text{Milk}), \ldots, (\text{Chicken}, \text{Slaw}, \text{Coffee}) \}$$

$$(\text{Chicken}, \text{Fruit}, \text{Soda}) \in \text{Main} \times \text{Side} \times \text{Drink}$$

$$A \times A \times A = A^3$$
$$A \times A \times A \times A = A^4$$
$$\vdots$$

Strings.

$A = \{0, 1, 11\}$

$11 = (11)$

$(1,1)?$

If A is a set of symbols, elements in A^n can be denoted by strings (without parens and commas).

Example: $\underline{A} = \underline{\{a, b\}}$.

Elements in $\underline{A^3}$ can be written as $\underline{abb}$ instead of (a, b, b).

What is $\{0, 1\}^3 = \{000, 001, 010, 011, 100, 101, 110, 111\}$

$D = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$L = \{a, b, c, \ldots, z\}$.

$6ctz \in D \times L \times L \times L = \underline{D \times L^3}$.

Strings of length 4 (digits or lower case letters):

$$(D \cup L)^4$$

$$(D \cup L) \times (D \cup L) \times (D \cup L) \times (D \cup L)$$

Length 2 or 3 starts w/ digit

$$(D \times (D \cup L)) \cup (D \times (D \cup L)^2)$$
$$= D \times ((D \cup L) \cup (D \cup L)^2).$$

Binary Strings

length 5 $\{0,1\}^5$

length 5 or 6 $\{0,1\}^5 \cup \{0,1\}^6$

length 5 starts w/ a 0:

$$\{0\} \times \{0,1\}^4$$

length 5, first + last bit are the same:

$$\{0\} \times \{0,1\}^3 \times \{0\} \cup \{1\} \times \{0,1\}^3 \times \{1\}$$

$B = \{0,1\}$

3 bit strings B^3

Set A:

Partition of A:
Set of subsets $A_1, \dots, A_n$

$A_i \cap A_j = \emptyset$

Each element $a \in A$ is an element of exactly one of the subsets.

$A_1 \cap A_2 \cdots \cap A_n = \emptyset$.

$A_1 \cup \cdots A_n = A$.

Example: A = Students enrolled in ICS 6B

S_a = students whose last name starts w/ a or A.

$\vdots$

S_z = students whose last names starts w/ z or Z.

$S_a, S_b, S_c, \dots, S_z$ form a partition of A.

Another example

$S_0 = \{ x \in \mathbb{N} :$ remainder of x divided by 3 is $0 \}$

$S_1 = \{ x \in \mathbb{N} :$ remainder of x divided by 3 is $1 \}$

$S_2 = \{ x \in \mathbb{N} :$ remainder of x divided by 3 is $2 \}$

S_1, S_2, S_3 form a partition of $\mathbb{N}$.

Functions

Note Title 10/10/2014

Continuous functions: $f(x) = x^2 - 3$
$f(x) = \sin^2(x)$

Discrete math: functions whose inputs & outputs are from finite or countably infinite sets.

Example: map a set of computational tasks to computers in a distributed network.

The input/output relationship of a digital circuit. (or computer program)

First part of definition: input & output sets.

$f: A \to B$ — A is the domain, B is the target (co-domain) } A & B are sets.

$f: \{0,1\}^5 \to \{0,1\}^4$ — input: binary strings length 5; output: binary strings of length 4.

$f(01101) = 1001$

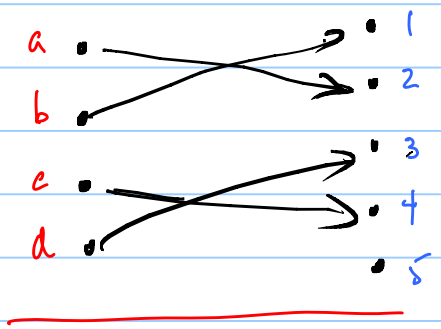
f must be well-defined for each element in the domain.
↳ one output in range for each input.

$f: \mathbb{R} \to \mathbb{R}$ → $f(x) = 1/x$ — NOT defined for $x=0$.

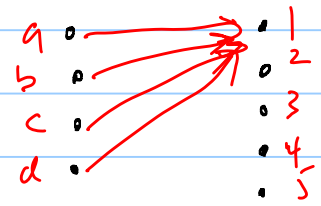
$f(x) = \pm\sqrt{x^2+1}$ — Two different outputs for each input.

Arrow diagram. $X = \{a, b, c, d\}$. $g: X \to Y$

$Y = \{1, 2, 3, 4, 5\}$.



$f(a) = 2$
$f(b) = 1$
$f(c) = 4$
$f(d) = 3$



Range of a function $f \subseteq$ Target. $f: X \to Y$

Range of $f = \{ y \in Y : \exists x \in X \;\; f(x) = y \}$.

For example above: $\{1, 2, 3, 4\}$

$f: \mathbb{R} \to \mathbb{R}$ $\quad f(x) = x^2$ $\quad$ range: non-neg reals.

$g: \mathbb{Z} \to \mathbb{Z}$ $\quad f(x) = 3x$ $\quad$ range: multiples of 3

$f: \{0,1\}^2 \to \{0,1\}^3$ $\quad f(x)$: copy the first bit and append to the end.

$\{00, 01, 10, 11\}$.

$f(00) = 000$
$f(01) = 010$
$f(10) = 101$
$f(11) = 111$

Range: $\{000, 010, 101, 111\}$

Range $\stackrel{?}{=}$ Target

NO.