Binary Relations.

Set $A \times B$

Binary Relation on/between $A \times B$ is a subset of $A \times B$.

$R \subseteq A \times B$

$(a,b) \in R \iff aRb.$

Example: Files in a large database stored in a distributed network.

$F = \text{set of all files}.$
$S = \text{set of all servers in the network}.$

Relation $R$ to express whether a particular file $f$ stored on a particular server $s$.

$fR_s \iff f \text{ stored on } s.$

There can be more than one copy of a particular file stored on different servers (for fault tolerance).

$R = \{(f_1, s_2), (f_1, s_2), (f_2, s_1), (f_3, s_3), (f_3, s_3), (f_4, s_2), (f_5, s_1), (f_5, s_2)\}$

$F = \{f_1, f_2, f_3, f_4, f_5\}$

$S = \{s_1, s_2, s_3\}$
A relation $R$ between sets $F + S$ is mathematically the same as a predicate with two input variables:

- $F$ domain of $x$ + $S$ domain of $y$.

For all $f \in F$ and $s \in S$, $R(f, s) = \text{true}$ if $(f, s) \in R$.

A relation is a generalization of a function.

Every function $f : F \rightarrow S$ is a relation between $F + S$.

Some relations between $F + S$ are not well-defined functions.

These are both relations but not functions.

A relation can be empty:
A relation on finite sets can be represented by a matrix:

\[
\begin{bmatrix}
1 & 1 & 0 \\
1 & 0 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 1 & 0 \\
\end{bmatrix}
\]

\[R \subseteq \mathbb{F} \times \mathbb{S}, \quad \text{rows} \rightarrow \text{elements in F}, \quad \text{columns} \rightarrow \text{elements in S}.
\]

Relations can be between infinite sets:

Relation \(C\) between \(\mathbb{Z}\) and \(\mathbb{R}\),

\[x \leq y \quad \text{iff} \quad |x-y| < 2. \quad \text{Sub set of} \quad \mathbb{Z} \times \mathbb{R}.
\]

\[\rightarrow 3 \leq 5.2 \quad \frac{|3-5.2| = |2.2| = 2.2 \leq 2.}{3 \leq 1.1} \quad \frac{|-4-(-5.5)| = |1.5| < 2.}{-4 \leq -5.5}
\]

\[\text{(()(()))} \]

\[\uparrow\]
Relation $R$ between a set $A$ and itself is a subset of $A \times A$.

"$R$ is a binary relation on the set $A$.

$A$ is the domain of $R$.

**Examples**: Relation $P$ on $\mathbb{N}$.

\[
\begin{align*}
aPb & \text{ if there is an } n \in \mathbb{N} \\
& \text{ such that } b = a^n \\
& \text{ "} b \text{ is a power of } n \text{."}
\end{align*}
\]

\[
\begin{align*}
3P9 & \text{ yes. } 3^2 = 9 \\
9P3 & \text{ no. } 9^2 = 3 \\
2P16 & \text{ yes. } 2^4 = 16 \\
2P20 & \text{ no.}
\end{align*}
\]

$A = \{a, b, c, d, e\}$

$R = \{(a, b), (b, b), (c, c), (c, e), (e, c), (c, d)\}$
Properties of Relations: Relation \( R \) on \( A \)

Reflexive: \( \forall a \in A, (a,a) \in R. \)

Anti-Reflexive: \( \forall a \in A, (a,a) \notin R. \)

\( S = \) a set of people.

Relation: \( M \) \( sMt \) if \( s \) and \( t \) have the same birthday.

Reflexive.

\( E \) \( sEt \) if \( s \) earns more money than \( t \).

Anti-reflexive.

\( M \) \( sMt \) if \( s \) sent an email to \( t \) yesterday.

Neither.

Relation \( R \) on \( \mathbb{R} \). \( xRy \) if \( |x| = y \).

\( 2 \overset{R}{\rightarrow} 2 \)? Neither.

\( -2 \overset{R}{\rightarrow} -2 \)?
Symmetric / Anti-Symmetric. Relation $R$ on $A$.

**Symmetric:** for every $x, y \in A$ \[ xRy \rightarrow yRx \]

For $x \neq y$:

- $\rightarrow$  
- $\leftarrow$

Never have:

$\rightarrow$ or $\leftarrow$

$xRy$ : $x$ is a sibling of $y$. Then $y$ is a sibling of $x$.

**Anti-Symmetric:** For every $x, y \in A$

- $xRy \land yRx \rightarrow x = y$
  
  equivalent to $xRy \land x \neq y \rightarrow yRx$

Can never have:

$\rightarrow$

(If $\forall a \in A$, $aR a$ then $a = a$)

\[ xMy \quad \text{x is a parent of y} \]
Anti-Symmetric!

| 2 \in \mathbb{Z} & \times \in \mathbb{R} & y \in \mathbb{N} & x^n = y. |
|-------------------|---------------------------------|
| 2 \times 2 = 4   & \iff 2 \times 4 = 8 |

**Reflexive.**

<table>
<thead>
<tr>
<th>x \in \mathbb{R} &amp; x \leq y.</th>
</tr>
</thead>
<tbody>
<tr>
<td>x \neq x \iff \text{anti-reflexive.}</td>
</tr>
</tbody>
</table>

**Anti-Symmetric.**

<table>
<thead>
<tr>
<th>E \in \mathbb{R} &amp; \times \in \mathbb{R} &amp; x \leq y.</th>
</tr>
</thead>
<tbody>
<tr>
<td>\iff x = y.</td>
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</table>

**Reflexive.**

<table>
<thead>
<tr>
<th>\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}</th>
</tr>
</thead>
<tbody>
<tr>
<td>\text{anti-Symmetric}</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>3 \times 3 = 9</th>
</tr>
</thead>
<tbody>
<tr>
<td>\text{not symmetric.}</td>
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**Neither reflexive or anti-reflexive.**

<table>
<thead>
<tr>
<th>2 \times 2 = 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>\text{not symmetric.}</td>
</tr>
</tbody>
</table>

<table>
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<tr>
<th>5 \times 5 = 25</th>
</tr>
</thead>
<tbody>
<tr>
<td>\text{not symmetric.}</td>
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</table>
A relation \( R \) on set \( S \) is transitive if

\[
xRy \text{ and } yRz \implies xRz.
\]

Never have:

\[
\begin{align*}
x & \rightarrow y \\
y & \rightarrow z \\
z & \not\rightarrow x
\end{align*}
\]

\[
\begin{align*}
x & \rightarrow b \\
\exists z & \quad y \rightarrow z \\
b & \rightarrow c \\
c & \rightarrow x
\end{align*}
\]

A reflexive, symmetric, and transitive relation is called an equivalence relation.

Let \( x, y, z \in \mathbb{Z} \):

\[
xPy \text{ if and only if } \exists n \in \mathbb{N} \quad x^n = y.
\]

Transitive:

\[
xPy \quad \text{and} \quad yPz \implies \quad xPz.
\]

\[
\begin{align*}
(x^n)^m &= z \\
x^{nm} &= z
\end{align*}
\]

\[
\begin{align*}
M & \quad \text{set if set have the} \\
& \quad \text{same birthday.}
\end{align*}
\]

M is not transitive.

\[
E & \quad \text{set if } s \text{ earns more money} \\
& \quad \text{than } t.
\]

E is not transitive.
On $\mathbb{Z}$, $a \mid b$ if $a$ and $b$ are relatively prime.

$\Rightarrow$ M sMt if $s$ sent an email to $t$ yesterday.

Not necessarily true.

On $\mathbb{Z}^+$, $a \mid b$ if $a$ evenly divides $b$.

$b = ak \quad k \in \mathbb{Z}^+$

$b$ is an integer multiple of $a$.

$2 \mid 20$ ? $\quad 2 \mid 60$

$\Rightarrow$ $2 \mid 60$.

$a \mid b$ and $b \mid c$ from $a \mid c$?