The next two questions pertain to the following recrusive definition of *full binary trees*. **Note: this is the same definition that was given in your homework.**

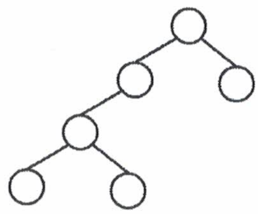
Basis: A single vertex with no edges is a full binary tree. The root is the only vertex in the tree.

Recursive rule: If T1 and T2 are full binary trees, then a new tree T' can be constructed by first placing T1 to the left of T2, adding a new vertex v at the top and then adding an edge between v and the root of T1 and an edge between v and the root of T2. The new vertex v is the root of T'.

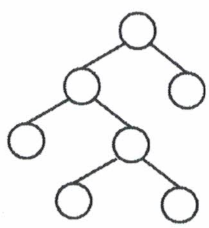
Exclusion statement: T is a full binary tree only if T is the tree described in the Basis or T can be constructed by repeatedly applying the recursive rule starting with the tree given in the basis.

5. Which of the following trees is **NOT** a full binary tree?

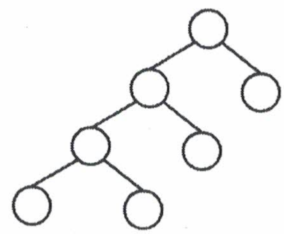
(a)



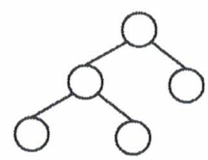
(b)



(c)



(d)



6. Suppose that a full binary tree T' is created using the recursive rule applied to full binary trees T1 and T2. Let $v(T)$ denote the number of vertices in tree T. Which expression correctly describes $v(T')$?

(a) $v(T') = v(T1) + v(T2)$

(b) $v(T') = 2 \cdot v(T1) + 2 \cdot v(T2) + 1$

(c) $v(T') = 2 \cdot v(T1) + 2 \cdot v(T2)$

(d) $v(T') = v(T1) + v(T2) + 1$

7. Select the summation expression that is equal to the following sum: $1^3 + 3^3 + 5^3 + \cdots + 99^3$.

(a)
$$\sum_{j=1}^{50} (2k+1)^3$$

(b)
$$\sum_{j=1}^{49} (2k+1)^3$$

(c)
$$\sum_{j=0}^{49} (2k+1)^3$$

(d)
$$\sum_{j=1}^{99} j^3$$

8. Select the expression that is equal to the following summation:

$$\sum_{j=1}^{m+3} j \cdot 2^{j-1}$$

(a) $\displaystyle\sum_{j=1}^{m+2} j \cdot 2^{j-1} + (m+3)2^{m+3}$

(c) $\displaystyle\sum_{j=1}^{m+2} j \cdot 2^{j-1} + (m+3)2^{m+2}$

(b) $\displaystyle\sum_{j=1}^{m+3} j \cdot 2^{j-1} + (m+3)2^{m+2}$

(d) $\displaystyle\sum_{j=1}^{m+2} j \cdot 2^{j-1} + (m+3)2^{m-1}$

The next three questions pertain to a proof by induction of the following theorem:
Theorem: For any integer $n \geq 4$, $2^n \geq 3n$
Here is the inductive step of the proof with some pieces missing:
Inductive step: Suppose that ____________________ then we will show that $2^{k+1} \geq 3(k+1)$.

$$\begin{aligned} 2^{k+1} &\geq 2 \cdot 2^k && \text{(Line 1)} \\ &\geq 2 \cdot 3k && \text{(Line 2)} \\ &\geq 3k + 3k && \text{(Line 3)} \\ &\geq 3k + 3 && \text{(Line 4)} \\ &\geq 3(k+1) && \text{(Line 5)} \end{aligned}$$

9. What is the correct statement to fill in the blank?

(a) for any $k \geq 4$, $2^k \geq 3k$,

(c) for any n, $2^n \geq 3n$,

(b) for any $k \geq 1$, $2^{k-1} \geq 3(k-1)$,

(d) for any $k \geq 1$, $2^k \geq 3k$,

10. Which line uses the fact that $k \geq 4 \geq 1$?

(a) Line 1

(b) Line 2

(c) Line 3

(d) Line 4

(e) Line 5

11. In which line is the inductive hypothesis used?

(a) Line 1

(b) Line 2

(c) Line 3

(d) Line 4

(e) Line 5

12. *(3 points)* The sequence $\{g_n\}$ is defined recursively as follows:

$$g_0 = 1 \qquad \text{and} \qquad g_n = 3 \cdot g_{n-1} + 2n, \quad \text{for } n \geq 1$$

Theorem 1. *For any non-negative integer n, $g_n = \frac{5}{2} \cdot 2^n - n - \frac{3}{2}$.*

If the theorem above is proven by induction, what must be established in the inductive step?

(a) For $k \geq 0$, if $g_k = \frac{5}{2} \cdot 2^k - k - \frac{3}{2}$, then $g_{k+1} = 3 \cdot g_k + 2(k+1)$.

(b) For $k \geq 0$, if $g_k = 3 \cdot g_{k-1} + 2k$, then $g_{k+1} = 3 \cdot g_k + 2(k+1)$.

(c) For $k \geq 0$, if $g_k = \frac{5}{2} \cdot 2^k - k - \frac{3}{2}$, then $g_{k+1} = \frac{5}{2} \cdot 2^{k+1} - (k+1) - \frac{3}{2}$.

(d) For $k \geq 0$, if $g_k = 3 \cdot g_{k-1} + 2k$, then $g_{k+1} = \frac{5}{2} \cdot 2^{k+1} - (k+1) - \frac{3}{2}$.

13. A sequence $\{a_n\}$ is defined as follows: $a_0 = 2$, $a_1 = 1$. For $n \geq 2$: $a_n = 3a_{n-1} - a_{n-2} + 1$. What is a_3?

(a) 17

(b) 7

(c) 6

(d) 2

14. Consider a proof by induction of a theorem that says, for any positive integer n, $Q(n)$ is true. What must be proven in the base case?

(a) $Q(k)$ is true

(b) $Q(0)$ is true

(c) $Q(1)$ is true

(d) $Q(n)$ is true

15. Consider a proof by induction of a theorem that says, for any positive integer n, $Q(n)$ is true. What must be proven in the inductive step?

(a) For any integer $k \geq 1$, $Q(k)$ implies $Q(n)$.

(b) For any integer $k \geq 1$, $Q(k)$ implies $Q(k+1)$.

(c) For any integer $k \geq 1$, $Q(k)$

(d) For any integer $k \geq 1$, $Q(k-1)$ implies $Q(k)$.

The next three questions pertain to the following recurrence relation:

- $f_0 = 3$
- $f_1 = 7$
- $f_n = 4 \cdot f_{n-1} - 4 \cdot f_{n-2}$

16. What is the characteristic equation for the recurrence relation?

(a) $x^2 + 4x - 4 = 0$

(b) $x^2 + 4x + 4 = 0$

(c) $x^2 - 4x + 4 = 0$

(d) $x^4 - 4x - 4 = 0$

17. What is the general solution for the recurrence relation?

(a) $f_n = \alpha_0 \cdot 2^n + \alpha_1 \cdot 4^n$

(b) $f_n = \alpha_0 \cdot 2^n + \alpha_1 \cdot n2^n$

(c) $f_n = \alpha_0 \cdot 2^n + \alpha_1 \cdot 2^n$

(d) $f_n = \alpha_0 \cdot 2^n + \alpha_1 \cdot (-2)^n$

18. Which two linear equations must be solved to fill in the correct constants α_0 and α_1 in the general solution?

(a) $3 = \alpha_1$
$7 = 2\alpha_0 - 2\alpha_1$

(b) $3 = \alpha_0 + \alpha_1$
$7 = 2\alpha_0 - 2\alpha_1$

(c) $3 = \alpha_0$
$7 = 2\alpha_0 + 2\alpha_1$

(d) $3 = \alpha_0 + \alpha_1$
$7 = 2\alpha_0 + 2\alpha_1$

(e) $3 = \alpha_0 + \alpha_1$
$7 = 2\alpha_0 + 4\alpha_1$

19. Suppose that in a proof by strong induction, the following statement is the inductive hypothesis:
For $k \geq 6$, $P(j)$ is true for any j in the range $\{4, 5, \ldots, k\}$.
Which of the following statements can you assume to be true?

(a) $P(k-2)$ and $P(k-1)$, but not $P(k-3)$

(b) Neither $P(k-3)$, $P(k-2)$, nor $P(k-1)$.

(c) $P(k-1)$, but not $P(k-2)$ or $P(k-3)$

(d) $P(k-3)$, $P(k-2)$, and $P(k-1)$

Recursive Algorithm

The function receives two inputs: a and n. a is a real number and n is an integer such that $n \geq 1$. It should return

$$SuperPower(a, n) = a^{4n-2}$$

Note that in the expression above, the exponent of a is $4n - 2$. Below is a recursive algorithm to compute SuperPower(a, n) with some lines missing.

```
SuperPower( a, n )
    If ( A ) Return( B )           // Base case
    y := SuperPower( C , D ) // Recursive Call
    Return( E )                    // Mathematatic expression using y and/or a
End
```

(2 points each question:)

20. For the recursive algorithm, what expression should go in the space labeled A?

(a) $a = 0$
(b) $n = 0$
(c) $n = 1$
(d) $a = 1$

21. For the recursive algorithm, what expression should go in the space labeled B?

(a) a^4
(b) a^2
(c) a
(d) 1

22. For the recursive algorithm, what expression should go in the space labeled C?

(a) a
(b) $n - 1$
(c) $a - 1$
(d) a DIV 2

23. For the recursive algorithm, what expression should go in the space labeled D?

(a) n
(b) $n - 4$
(c) n DIV 2
(d) $n - 1$

24. For the recursive algorithm, what expression should go in the space labeled E?

(a) $y^4 \cdot a$
(b) y^4
(c) a^4
(d) $y \cdot a^4$