Note Title 6/3/2013

In this lecture we'll give more background on the relationship between hardness of approximation and PCP.

The Constraint Satisfaction Problem is a generalization of SAT:

CSP: q is a natural number. qCSP instance is a collection of functions $\varphi_1 \cdots \varphi_m$ (called constraints). $\varphi_i : \{0,1\}^n \to \{0,1\}$. Each φ_i depends on at most q of the input locations.

$\forall i \in [m]$, $\exists$ $j_1 \cdots j_q \in [n]$ $f : \{0,1\}^q \to \{0,1\}$

$\varphi_i(u) = f(u_{j_1} \cdots u_{j_q})$ for every $u \in \{0,1\}^n$

u satisfies the i^{th} constraint φ_i if $\varphi_i(u) = 1$.

The fraction of constraints satisfied is $\dfrac{\sum_i \varphi_i(u)}{m}$

$\text{val}(\varphi)$ is the max of this value over all u.

φ is satisfiable if $\text{val}(\varphi) = 1$.

q is the arity of φ.

Gap CSP $\forall q \in \mathbb{N}$, $\rho \le 1$, define ρ-GAPqCSP to be the problem of determining for φ, instance of qCSP, whether

1) $\text{val}(\varphi) = 1$ YES instance
2) $\text{val}(\varphi) < \rho$ NO instance

ρ-GAPqCSP is NP-hard if for every $L \in NP$ $\exists$ poly-time computable function f

Completeness: $x \in L \Rightarrow \text{val}(f(x)) = 1$.

Soundness: $x \notin L \Rightarrow \text{val}(f(x)) < \rho$.

Theorem 1: $\exists\, q \in \mathbb{N}$, $\rho \in (0,1)$ s.t. ρ-GAP$_q$ CSP is NP-hard.

Theorem 2: NP = PCP $(\log n, 1)$.

Theorem 2 $\Rightarrow$ Theorem 1:

Assume NP $\subseteq$ PCP $(\log n, 1)$. Will show $1/2$-GAP$_q$ CSP is NP-hard for some p.

SAT has PCP system in which the verifier uses $c \cdot \log n$ random bits + queries q locations.

$V_{x,r}$ is the function that on input $\pi \in \{0,1\}^q$ is $= 1$ iff the verifier accepts.

$\varphi_x = \{V_{x,r}\}_{r \in \{0,1\}^{c\log n}}$ is a poly-sized instance of CSP.

Since V runs in poly-time the transformation from x to φ_x is also poly-time.

if $x \in$ SAT $\Rightarrow$ val $(\varphi) = 1$

$x \notin$ SAT $\Rightarrow$ val $(\varphi) \le 1/2$.

Theorem 1 $\Rightarrow$ Theorem 2:

ρ-GAP$_q$CSP is NP-hard for some q, $\rho < 1$.

$\Rightarrow$ translates to a PCP system w/ q queries, ρ soundness and log randomness. for any $L \in$ NP.

Run reduction of L to ρ-GAP$_q$CSP to get $\varphi = f(x)$.

Suppose $\varphi = \{\varphi_i\}_{1 \le i \le m}$.

The verifier will expect the proof to be an assignment to the variables.

Verifier selects random $i \in [m]$ and asks to see the variables involved in ϕ_i. (There are q).

if $x \in L \Rightarrow$ verifier accepts w.p. 1

if $x \notin L \Rightarrow$ verifier accepts w.p $\leq \rho$.

↳ Can be boosted to 1/2 at the expense of a constant factor in randomness and queries. //

What about hardness of approximation for other problems? We already showed a 1/2-approx scheme for vertex cover. Is it possible to do any better?

We will show that for VC there is a γ s.t. approximating VC to within γ is NP-hard. Also for any $\gamma < 1$ approximating Indep Set to within γ is NP-hard.

Note that the complement of a VC is an IS. So in terms of exact solutions, they have the same complexity.

Let VC be the size of the smallest vertex cover.

Let IS be the size of the largest independent set.

$$VC = n - IS.$$

A ρ-approx for Indep Set produces an independent of size at least $\rho \cdot IS$. What kind of approx factor does this give for VC? $\dfrac{n - IS}{n - \rho \cdot IS}$ this could be very small if IS is close to n.

We know that VC has a $1/2$-approx and IS can have no such approx alg — so approximability of the two problems ~~Bon~~ is very different.

First we will show that each problem has a ρ s.t. it can not be ρ-approximated. Then we will show how to amplify this factor for Ind Set.

Lemma: $\exists$ poly-time reduction from 3CNF $\to$ graphs s.t. $\forall \varphi$, $f(\varphi)$ is an n vertex graph whose largest indep set has size $\mathrm{val}(\varphi)\, n/7$.

└ for VC has size $\frac{6}{7} n\, \mathrm{val}(\varphi)$

This uses the normal reduction for Ind Set.
(each clause has a 7 node cluster which is a clique)
~~The~~ Each node represents an assignment that satisfies that clause. Then put edges between partial assignments in different clauses that are in conflict

~~x y z~~

x=T — u=T
y=F — w=F
z=F — z=T

Corollary: $\exists\ \rho < 1,\ \rho' < 1$

s.t. Ind Set can not be ρ-approximated in poly time & Vert Cover can not be ρ'-approx in poly time unless P=NP.

L $\in$ NP. $\exists\ f\ f'$

$x \in L \longrightarrow \varphi$ satisfiable $\longrightarrow G$ has Ind Set of size $\frac{m}{7}$

$x \notin L \longrightarrow \le \rho m$ clauses in φ satisfiable $\longrightarrow$ Ind set has size $< \frac{\rho m}{7}$

approximating IS to within ρ solves $x \overset{?}{\in} L$.

$x \in L \longrightarrow \varphi$ sat $\longrightarrow VC \leq \frac{6}{7}m$

$x \notin L \longrightarrow \leq \rho\frac{m}{7}$ clauses sat $\longrightarrow VC \geq \left(1 - \frac{\rho}{7}\right)m$

approx to within $\frac{6}{7-\rho}$ not possible unless $P = NP$.

Boosting for IS:

Graph $G \longrightarrow G^k$

$\llcorner$ vertices = size k subsets of vertices in G.

two vertices are not adj iff $S_1 \cup S_2$ is an Ind Set in G.

$IS \longleftrightarrow \binom{IS}{k}$.

Approx ratio for ρ in G $= \dfrac{\binom{\rho IS}{k}}{\binom{IS}{k}} \approx \rho^k$.

ρ^k can be made smaller than any const.

running time of reduction is $O(n^k)$ k constant.