SETS

Sections 2.1, 2.2 and 2.4

Chapter Summary

- Sets
  - The Language of Sets
  - Set Operations
  - Set Identities

Introduction

- Sets are one of the basic building blocks for the types of objects considered in discrete mathematics.
  - Important for counting.
  - Programming languages have set operations.
- Set theory is an important branch of mathematics.
  - Many different systems of axioms have been used to develop set theory.
  - Here we are not concerned with a formal set of axioms for set theory. Instead, we will use what is called naïve set theory.

Sets

- A *set* is an unordered collection of objects.
  - the students in this class
  - the chairs in this room
- The objects in a set are called the *elements*, or *members* of the set. A set is said to *contain* its elements.
- The notation $a \in A$ denotes that a is an element of the set A.
- If a is not a member of A, write $a \notin A$

Describing a Set: Roster Method

- $S = \{a,b,c,d\}$
- Order not important

  $$S = \{a,b,c,d\} = \{b,c,a,d\}$$
- Each distinct object is either a member or not; listing more than once does not change the set.

  $$S = \{a,b,c,d\} = \{a,b,c,b,c,d\}$$
- Elipses (…) may be used to describe a set without listing all of the members when the pattern is clear.

  $$S = \{a,b,c,d, \ldots\ldots,z\}$$

Roster Method

- Set of all vowels in the English alphabet:

$$V = \{a,e,i,o,u\}$$

- Set of all odd positive integers less than 10:

$$O = \{1,3,5,7,9\}$$

- Set of all positive integers less than 100:

$$S = \{1,2,3,\ldots\ldots..,99\}$$

- Set of all integers less than 0:

$$S = \{\ldots., -3,-2,-1\}$$

Some Important Sets

$\mathbf{N}$ = *natural numbers* = $\{0,1,2,3\ldots.\}$

$\mathbf{Z}$ = *integers* = $\{\ldots,-3,-2,-1,0,1,2,3,\ldots\}$

$\mathbf{Z}^{+}$ = *positive integers* = $\{1,2,3,\ldots..\}$

$\mathbf{R}$ = set of *real numbers*

$\mathbf{R}^{+}$ = set of *positive real numbers*

$\mathbf{C}$ = set of *complex numbers*.

Q = set of rational numbers

Set-Builder Notation

- Specify the property or properties that all members must satisfy:

  $S = \{x \mid x \text{ is a positive integer less than } 100\}$

  $O = \{x \mid x \text{ is an odd positive integer less than } 10\}$

  $O = \{x \in \mathbf{Z}^{+} \mid x \text{ is odd and } x < 10\}$

- A predicate may be used:

  $$S = \{x \mid \mathrm{P}(x)\}$$

- Example: $S = \{x \mid \mathrm{Prime}(x)\}$
- Positive rational numbers:

  $$\mathbf{Q}^{+} = \{x \in \mathbf{R} \mid x = p/q, \text{ for some positive integers } p, q\}$$

Interval Notation

$$[a,b] = \{x \mid a \le x \le b\}$$

$$[a,b) = \{x \mid a \le x < b\}$$

$$(a,b] = \{x \mid a < x \le b\}$$

$$(a,b) = \{x \mid a < x < b\}$$

closed interval [a,b]

open interval (a,b)

Universal Set and Empty Set

- The *universal set U* is the set containing everything currently under consideration.
  - Sometimes implicit
  - Sometimes explicitly stated.
  - Contents depend on the context.
- The empty set is the set with no elements. Symbolized ∅, but {} is also used.

Subsets

Definition: The set A is a *subset* of B, if and only if every element of A is also an element of B.

- The notation $A \subseteq B$ is used to indicate that A is a subset of the set B.
- $A \subseteq B$ holds if and only if $\forall x(x \in A \rightarrow x \in B)$ is true.

Because $a \in \emptyset$ is always false, $\emptyset \subseteq S$,for every set S.

Because $a \in S \rightarrow a \in S$, $S \subseteq S$, for every set S.

Showing a Set is or is not a Subset of Another Set

- **Showing that A is a Subset of B**: To show that $A \subseteq B$, show that if x belongs to A, then x also belongs to B.
- **Showing that A is not a Subset of B**: To show that A is not a subset of B, $A \nsubseteq B$, find an element $x \in A$ with $x \notin B$. (Such an x is a counterexample to the claim that $x \in A$ implies $x \in B$.)

Examples:

1. The set of all computer science majors at your school is a subset of all students at your school.

Another look at Equality of Sets

- Recall that two sets *A* and *B* are *equal*, denoted by *A* = *B*, iff $\forall x(x \in A \leftrightarrow x \in B)$

- Using logical equivalences we have that *A* = *B* iff

$$\forall x[(x \in A \rightarrow x \in B) \wedge (x \in B \rightarrow x \in A)]$$

- This is equivalent to

$A \subseteq B$ and $B \subseteq A$

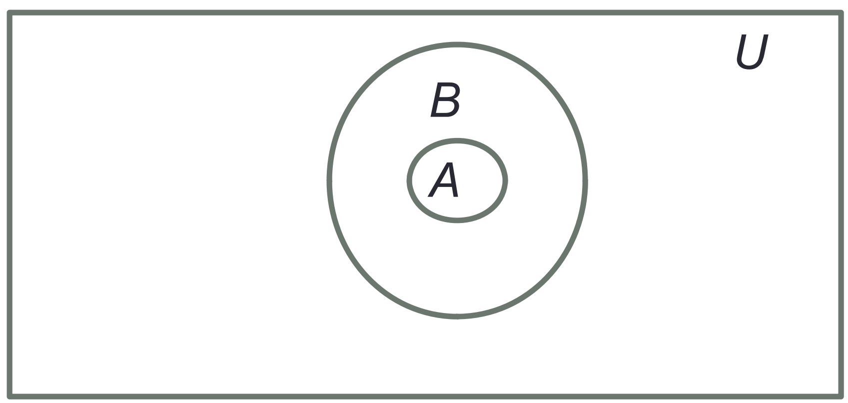
Proper Subsets

Definition: If $A \subseteq B$, but $A \neq B$, then we say A is a *proper subset* of B, denoted by $A \subset B$. If $A \subset B$, then

$$\forall x(x \in A \rightarrow x \in B) \wedge \exists x(x \in B \wedge x \notin A)$$

is true.

Venn Diagram



Set Cardinality

Definition: If there are exactly n distinct elements in *S* where *n* is a nonnegative integer, we say that *S* is *finite*. Otherwise it is *infinite*.

Definition: The *cardinality* of a finite set *A*, denoted by |*A*|, is the number of (distinct) elements of *A*.

Examples:

$|\varnothing| = 0$

Let S be the letters of the English alphabet. Then $|S| = 26$

$|\{1,2,3\}| = 3$

The set of integers is infinite.

Power Sets

Definition: The set of all subsets of a set *A*, denoted P**(***A***)**, is called the *power set* of *A*.

Example: If *A* = {a,b} then

$$\mathcal{P}(A) = \{\emptyset, \{a\},\{b\},\{a,b\}\}$$

- If a set has *n* elements, then the cardinality of the power set is 2^n.
  - *If B = {1, 2, 3, 4}*
  - $|B|=4$
  - $|\mathcal{P}(B)| = 2^4=16$
  - $\{1,2,4\} \subseteq B$ *and* $\{1,2,4\} \in \mathcal{P}(B)$
  - $\{4\} \subseteq B$ *and* $\{4\} \in \mathcal{P}(B)$
  - $4 \in B$ but $4 \notin \mathcal{P}(B)$

Union

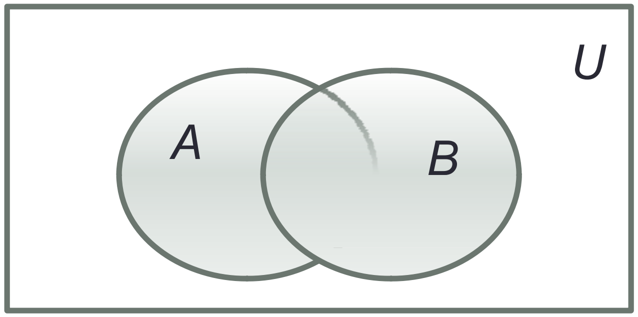
- **Definition**: Let *A* and *B* be sets. The *union* of the sets *A* and *B*, denoted by $A \cup B$, is the set:

$$\{x | x \in A \vee x \in B\}$$

- **Example**: What is $\{1,2,3\} \cup \{3, 4, 5\}$?

Solution: $\{1,2,3,4,5\}$

Venn Diagram for $A \cup B$



Intersection

- **Definition**: The *intersection* of sets *A* and *B*, denoted by $A \cap B$, is $\{x \mid x \in A \wedge x \in B\}$

- Note if the intersection is empty, then *A* and *B* are said to be *disjoint*.
- **Example**: What is? $\{1,2,3\} \cap \{3,4,5\}$?

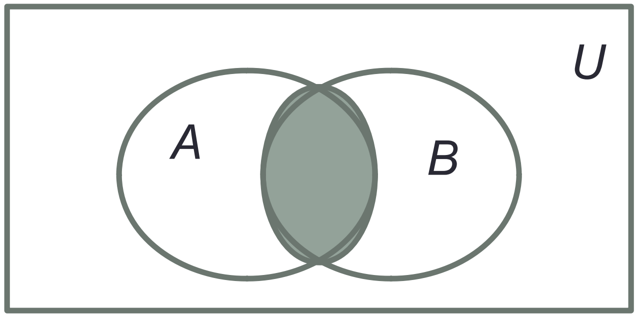
  Solution: $\{3\}$

- **Example:** What is?

  $\{1,2,3\} \cap \{4,5,6\}$?

  Solution: $\emptyset$

Venn Diagram for $A \cap B$



Complement

Definition: If A is a set, then the complement of the A (with respect to U), denoted by $\bar{A}$ is the set $U - A$

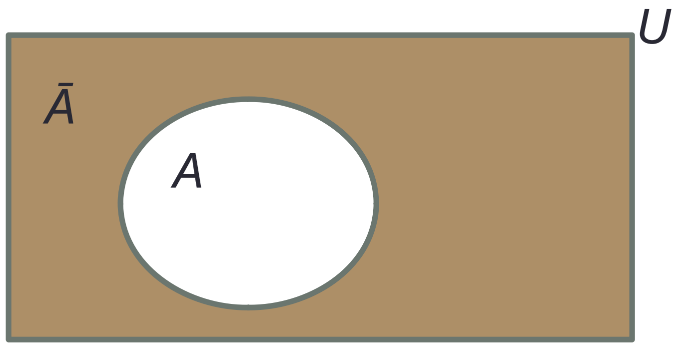
$$\bar{A} = \{x \in U \mid x \notin A\}$$

(The complement of A is sometimes denoted by A^c .)

Example: If U is the positive integers less than 100, what is the complement of $\{x \mid x > 70\}$

Solution: $\{x \mid x \leq 70\}$

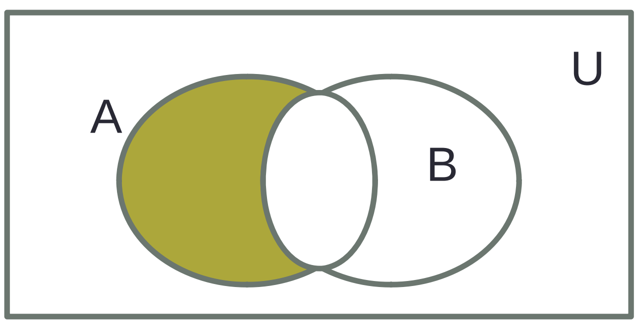
Venn Diagram for Complement



Difference

- **Definition**: Let *A* and *B* be sets. The *difference* of *A* and *B*, denoted by *A* – *B*, is the set containing the elements of *A* that are not in *B*. The difference of *A* and *B* is also called the complement of *B* with respect to *A*.

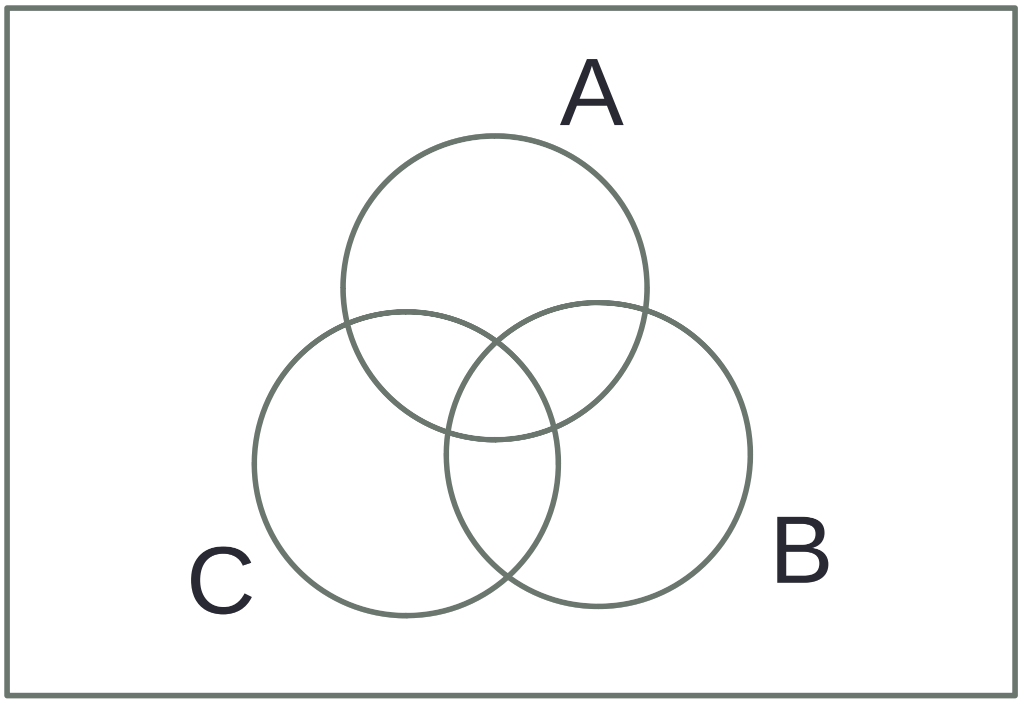
$$A - B = \{x \mid x \in A \wedge x \notin B\} = A \cap \overline{B}$$



Venn Diagram for $A - B$

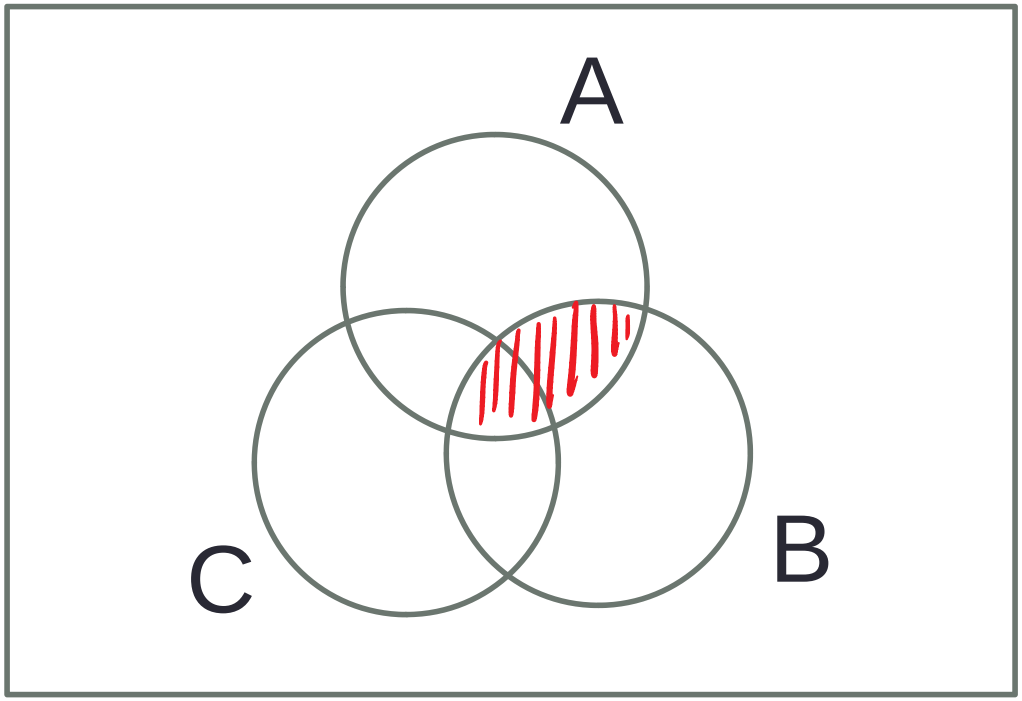
Combining set operations

$(A \cap B) \cup C$



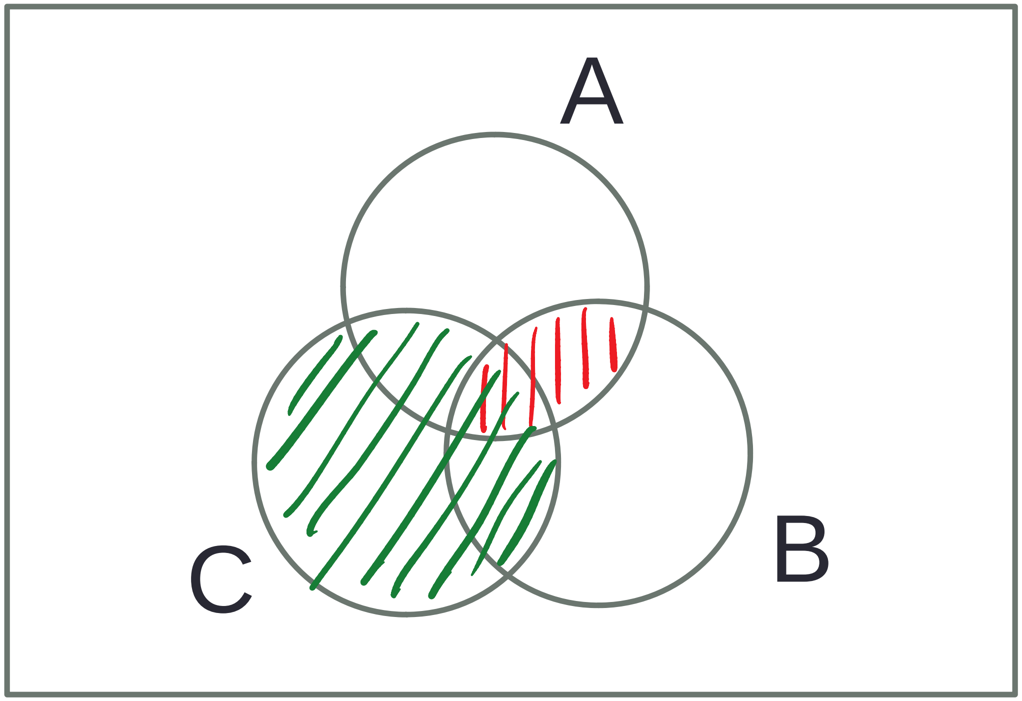
Combining set operations

$(A \cap B) \cup C$



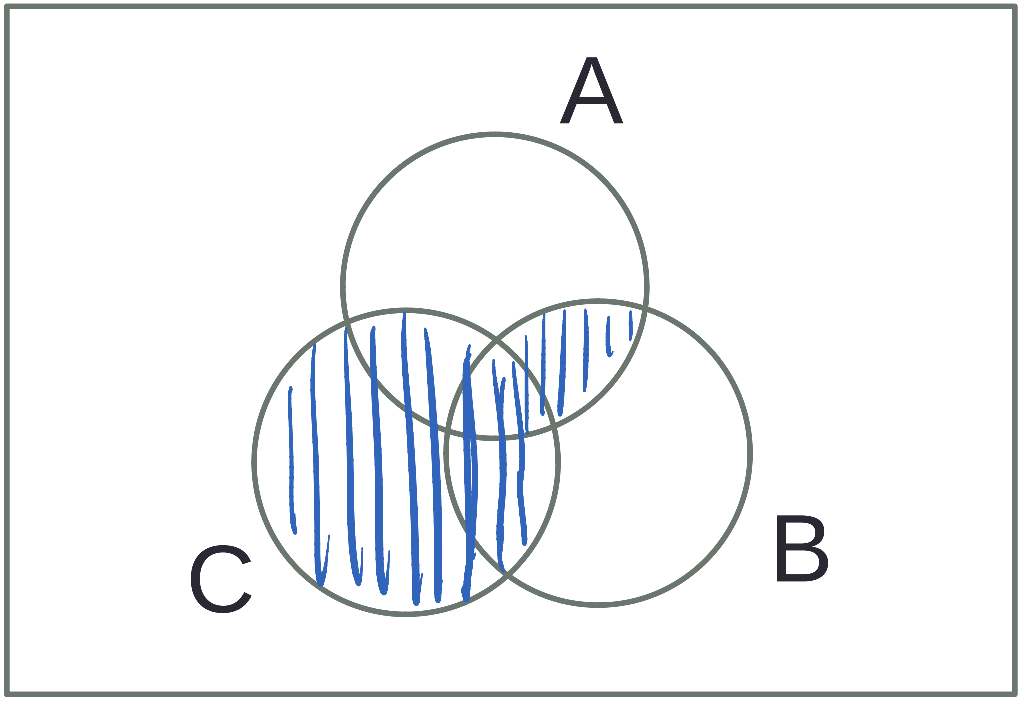
Combining set operations

$(A \cap B) \cup C$



Combining set operations

$(A \cap B) \cup C$



Review Questions

Example: $U = \{0,1,2,3,4,5,6,7,8,9,10\}$ $A = \{1,2,3,4,5\}$, $B = \{4,5,6,7,8\}$

1. $A \cup B$

   Solution: $\{1,2,3,4,5,6,7,8\}$

2. $A \cap B$

   Solution: $\{4,5\}$

3. $\bar{A}$

   Solution: $\{0,6,7,8,9,10\}$

4. $\bar{B}$

   Solution: $\{0,1,2,3,9,10\}$

5. $A - B$

   Solution: $\{1,2,3\}$

6. $B - A$

   Solution: $\{6,7,8\}$

Set Identities

- Identity laws $A \cup \emptyset = A$ $A \cap U = A$
- Domination laws $A \cup U = U$ $A \cap \emptyset = \emptyset$
- Idempotent laws $A \cup A = A$ $A \cap A = A$
- Complementation law $\overline{(\overline{A})} = A$

Continued on next slide →

Set Identities

- Commutative laws $A \cup B = B \cup A \quad A \cap B = B \cap A$

- Associative laws
$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

- Distributive laws
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Continued on next slide →

Set Identities

- De Morgan's laws

$$\overline{A \cup B} = \overline{A} \cap \overline{B} \qquad \overline{A \cap B} = \overline{A} \cup \overline{B}$$

- Absorption laws

$$A \cup (A \cap B) = A \qquad A \cap (A \cup B) = A$$

- Complement laws

$$A \cup \overline{A} = U \qquad A \cap \overline{A} = \emptyset$$

Set Partitions

- A partition of set A is a set of sets $B_1,..,B_n$
  - Each B_i is non-empty for i ∈ $\{1, \dots, n\}$
  - Each pair is disjoint: $B_i \cap B_j \neq \emptyset$
    - for i, j ∈ $\{1, \dots, n\}$
  - Their union is A:
    - $B_1 \cup .. \cup B_n = A$

  - Example: partition of $\mathbb{R}$
    - $(\infty, -2]$
    - (-2,3]
    - (3, ∞)