Generating Permutations and Subsets

ICS 6D

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Lexicographic Order

- S a set
- S^n is the set of all n-tuples whose entries are elements in S.

- If S is ordered, then we can define an ordering on the n-tuples of S called the ***lexicographic*** or ***dictionary*** order.

- For simplicity, we will discuss n-tuples of natural numbers.

Lexicographic Order: Example

(3, 8, 3, 4, 2, 1) <? Or >? (3, 8, 3, 2, 2, 1)

Lexicographic Order: Definition

$(p_1, p_2, \ldots, p_n) \neq (q_1, q_2, \ldots, q_n)$

Let k be the smallest index such that $p_k \neq q_k$

(If the n-tuples are different they have to differ somewhere).

If $p_k < q_k$, then $(p_1, p_2, \ldots, p_n) < (q_1, q_2, \ldots, q_n)$

If $p_k > q_k$, then $(p_1, p_2, \ldots, p_n) > (q_1, q_2, \ldots, q_n)$

Lexicographic Order: Examples

(2, 5, 100, 2, 4) (2, 5, 100, 2, 5)

(1, 100, 1000) (2, 1, 0)

(5, 4, 5, 6, 7) (4, 5, 8, 10, 11)

Generating all Permutations

- A permutation of {1, 2, …, n} is an n-tuple in which each number in {1, 2, …, n} appears exactly once.

  Example: (5, 2, 1, 6, 7, 4, 3) is a permutation of {1, 2, 3, 4, 5, 6, 7}

- How to generate all permutations over the set {1, 2, …, n}?

  Will output the permutations in lexicographic order

Lexicographic Order of Permutations

- Here are all the permutations of {1, 2, 3, 4} in lexicographic order:

(1, 2, 3, 4)	(1, 2, 4, 3)	(1, 3, 2, 4)	(1, 3, 4, 2)
(1, 4, 2, 3)	(1, 4, 3, 2)	(2, 1, 3, 4)	(2, 1, 4, 3)
(2, 3, 1, 4)	(2, 3, 4, 1)	(2, 4, 1, 3)	(2, 4, 3, 1)
(3, 1, 2, 4)	(3, 1, 4, 2)	(3, 2, 1, 4)	(3, 2, 4, 1)
(3, 4, 1, 2)	(3, 4, 2, 1)	(4, 1, 2, 3)	(4, 1, 3, 2)
(4, 2, 1, 3)	(4, 2, 3, 1)	(4, 3, 1, 2)	(4, 3, 2, 1)

Lexicographic Order of Permutations

- The smallest permutation of {1, 2,.., n} is

  (1, 2, 3,…, n)

- The largest permutation of {1, 2,.., n} is

  (n, n-1,…, 2, 1)

Lexicographic Order of Permutations

GeneratePermutationsInLexOrder(n)

```
Initialize P = (1, 2, …, n)
Output P
While P ≠ (n, n-1,.., 2, 1)
    P = GetNext(P)
    Output P
```

Get Next Permutation

(8, 2, 5, 3, 7, 6, 4, 1)

Get Next Permutation

GetNext $(p_1, p_2, \ldots, p_n)$

- Find the largest k such that $p_k < p_{k+1}$
- Find the largest j such that $j > k$ and $p_j > p_k$
- Swap p_j and p_k
- Reverse the order of $p_{k+1},\ldots,p_n$

Get Next Permutation

- Examples:

(2, 1, 4, 3, 5, 6)

(5, 2, 6, 4, 3, 1)

Get Next Permutation

- Examples:

(4, 5, 6, 3, 2, 1)

(6, 5, 4, 3, 2, 1)

r-Subsets

- The order in which the elements of a subset are listed does not matter, so

  {5, 8, 2} = {2, 5, 8}

In order to avoid over-counting subsets, we will always list their elements in ***increasing*** order:

Examples: {2, 5, 8}

{1, 4, 5, 16}

Lexicographic Order of r-Subsets

- First order the elements of the subset in increasing order
- The apply lexicographic ordering as if they were ordered subsets:

  {4, 1, 7, 3} {5, 4, 3, 2}

Lexicographic Order of r-Subsets

- What's the smallest 5-subset of

  {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13}?

- What's the largest 5-subset of

  {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13}?

Lexicographic Order of r-subsets

- Here are all the 3-subsets of {1, 2, 3, 4, 5} listed in lexicographic order:

{1, 2, 3}	{1, 2, 4}	{1, 2, 5}	{1, 3, 4}	{1, 3, 5}
{1, 4, 5}	{2, 3, 4}	{2, 3, 5}	{2, 4, 5}	{3, 4, 5}

Lexicographic Order of r-Subsets

Generate-r-SubsetsInLexOrder(r, n)

```
Initialize P = (1, 2, …, r)
Output P
While P ≠ (n-r+1, n-r+2,.., n-1, n)
        P = GetNext(P)
        Output P
```

Get Next r-Subset

{3, 4, 7, 8, 9}   from set {1, 2, 3, 4, 5, 6, 7, 8, 9}

Get Next r-Subset

GetNext $\{p_1, p_2, \ldots, p_r\}$

- Find the largest k such that $p_k < n-r+k$
- $p_k = p_k + 1$
- For $j = k+1,\ldots,r$

  $p_j = p_{j-1} + 1$

Get Next r-Subset

- Examples: (r = 5, n = 9)

{1, 2, 4, 5, 6}

{2, 4, 6, 7, 9}

Get Next r-Subset

- Examples: (r = 5, n = 9)

{2, 3, 5, 8, 9}

{4, 6, 7, 8, 9}