Binomial Coefficients and Combinatorial Identities

ICS 6D
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• Multiply the following polynomial:

\[(x + y)(x + y) =\]

\[(x + y)^3 = (x + y)(x + y)^2\]
\[(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3\]

To get the coefficient of the \(x^2y\) term:

- \(xxy\)
- \(xyx\)
- \(xyx\)
- \(yxx\)
- \(yxx\)
To generalize....

• \((x + y)^n = \text{sum over } 2^n \text{ terms, each of which is a } \text{“string” of length } n \text{ over } \{x, y\}\)

Coefficient of \(x^k y^{n-k}\) = the number of strings of length \(n\) with \(k\) \(x\)’s and \(n-k\) \(y\)’s
The Binomial Theorem

• For any $x$ and $y$, and any natural number $n$

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$
\[(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}\]

Apply to \((x + y)^5\)
\((x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}\)

Apply to \((3a - 2b)^6\)
\[(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}\]

Apply to \((-4a + 3b)^9\)
Binomial Theorem for Identities

\[(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}\]

Plug in \(x = y = 1\)
Combinatorial Argument for Identities

\[ 2^n = \sum_{k=0}^{n} \binom{n}{k} \]

Number of subsets of \{1, 2, 3, \ldots, n\}

\[ = \sum_{k=0}^{n} \text{Number of } k \text{-subsets of } \{1, 2, 3, \ldots, n\} \]
Pascal’s Identity

\[
\text{# k-subsets of } \{1, 2, \ldots, n, n+1\} = \text{# k-subsets of } \{1, 2, \ldots, n, n+1\} \text{ that do not include 1} + \text{# k-subsets of } \{1, 2, \ldots, n, n+1\} \text{ that DO include 1}
\]

**Example:** \( n = 4, k = 3 \)

3-subsets from \( \{1, 2, 3, 4, 5\} \)

\[\{2, 3, 4\}, \{2, 3, 5\}, \{2, 4, 5\}, \{3, 4, 5\}\]

\[\{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\}, \{1, 3, 4\}, \{1, 3, 5\}, \{1, 4, 5\}\]
Pascal’s Identity

\[
\text{# k-subsets of } \{1, 2, \ldots, n, n+1\} = \text{# k-subsets of } \{1, 2, \ldots, n, n+1\} \text{ that do not include 1} + \text{# k-subsets of } \{1, 2, \ldots, n, n+1\} \text{ that DO include 1}
\]
Pascal’s Triangle