Student ID Number:

Name:______________________________

Test I

Version A

ICS 6D

Winter 2017

February 3, 2017

Instructor: Sandy Irani

Instructions

- Wait until instructed to turn over the cover page.
- The total number of points on the test is 44.
- **Important:** Except for the cover page, there are questions on both sides of the page.

1. *(9 points)* Fill in the missing statements of the inductive proof below. You can add as many lines as you need. Make sure and label where you use the inductive hypothesis in your argument.

Theorem 1. *For any integer $n \geq 1$, 7 evenly divides* $6^{2n} - 1$.

Proof:

Base Case: ____

Inductive Step:

We will assume that 7 evenly divides $6^{2k} - 1$,

and prove that ____.

Since, by the inductive hypothesis, 7 evenly divides $6^{2k} - 1$, then 6^{2k} can be expressed as:

____, where m is an integer.

Then:

$$6^{2(k+1)} - 1 = \underline{\hspace{10cm}}$$

$$= \underline{\hspace{10cm}}$$

$$= 7 \cdot (\underline{\hspace{6cm}})$$

Since $6^{2(k+1)} - 1$ is an integer multiple of 7, then 7 evenly divides $6^{2(k+1)} - 1$. ■

2. *(8 points)* A sequence $\{f_n\}$ is defined by the following recurrence relations and initial conditions.

   - $f_0 = 5$
   - $f_1 = 15$
   - $f_n = 4 \cdot f_{n-1} + 21 \cdot f_{n-2}$

   Solve the recurrence relation. Please show all your steps so that we can give partial credit if you make a mistake.

3. *(2 points)* Suppose that the characteristic equation for a recurrence relation is

$$(x+1)(x-5)^2(x+6) = 0.$$

   Give the general solution for the recurrence relation.

4. *(8 points)* A function receives two inputs: a and n, where a is a real number and n is a non-negative integer and returns the value

$$\text{ExpPower}(a, n) = a^{3^n}.$$

Note that in the expression above, the exponent of a is 3^n. Fill in the blanks for recursive algorithm to compute ExpPower(a, n). The value returned should be a mathematical expression that only uses addition or multiplication operations and uses one or more of the variables y, a, n.

ExpPower(a, n)

If (________________) Return(________________) // Base case

y := ExpPower(__________, __________) // Recursive Call

Return(____________________________________)

End

5. *(4 points)* The function below receives two inputs: a and n, where a is a real number and n is a non-negative integer. The algorithm returns a^{2n}. (The exponent of a is $2n$.)

TwoPower(a, n)

If ($n = 0$) Return(1)

y := TwoPower(a, $n - 1$)

Return($y \cdot a \cdot a$)

End

Let NumMult(a, n) be the number of multiplication operations performed by the algorithm TwoPower on inputs a and n.

(a) What is NumMult$(a, 0)$?

(b) Express NumMult$(a, k + 1)$ as a function of NumMult(a, k).

6. *(4 points)* Express the following sums using summation notation (with out the ...).

   (a) $2^3 + 2^4 + 2^5 + \cdots 2^{22}$.

   (b) $4^3 + 6^3 + 8^3 + \cdots (24)^3$.

7. *(2 points)* Write down an equivalent expression to the summation below where the last term is outside the summation:

$$\sum_{j=-2}^{n-3} j^{j+4}$$

8. *(3 points)* Let $S(n)$ be a statement parameterized by a positive integer n. Consider a proof that uses strong induction to prove that for all $n \geq 4$, $S(n)$ is true. The base case proves that $S(4)$, $S(5)$, $S(6)$, $S(7)$, and $S(8)$ are all true. Fill in the blanks below to get a correct statement of what is assumed and proven in the inductive step.

   For $k \geq$____________, suppose that S(j) is true for every j in the range ____________ through k.

   We will show that ____________ is true.

9. *(4 points)* The sequence $\{g_n\}$ is defined recursively as follows:

$$g_0 = 1 \qquad \text{and} \qquad g_n = 3 \cdot g_{n-1} + 2n, \text{ for } n \geq 1$$

   Theorem 2. *For any non-negative integer n, $g_n = \frac{5}{2} \cdot 2^n - n - \frac{3}{2}$.*

   If the theorem above is proven by induction, fill in the blanks below to express what must be established in the inductive step. Fill in each blank with a mathematical equation (i.e., no English).

   For $k \geq 0$, if ____________________________________,

   then ____________________________________.

This area is for scratch work.