Announcements:

- We will start using R Commander again
- Friday discussion is for credit.

Homework: (Due Wed, Feb 20)

See handout on website, in daily calendar.
Use R Commander or Excel. Instructions on using R Commander for binomial probabilities are on the website in the R Commander section (and in today's lecture). Instructions using Excel are in the book (p. 278).

Today:

- Section 8.4 (binomial)
- Power point on research on psychics, if time

Section 8.4: Binomial Random Variables

What do the following random variables have in common?

Example 1: A fair coin is flipped 10 times,
X = number of heads.

Example 2: Ten births are observed at a hospital,
X = number of boys. For simplicity, assume P(Boy)=.5

Example 3: A student takes a 10 question true/false test, just guessing, X = number correct.

Example 4: Suppose *half* of all adults think genetically modified food is unsafe. Take a random sample of 10 adults, X = number (out of the 10 polled) who think this.

What do those random variables all have in common?

Each of these random variables has the exact same probability distribution function!

$P(X = 0) = (½)^{10}$ [Ex 1: X = 0 heads => TTTTTTTTTT]
P(X = 1) is the same for all of them, and so on.
Note that X can be 0, 1, 2, ... , 10

In each case, X is called a *binomial random variable with n=10 and p=½.*

It is the outcome of a *binomial experiment*.

Properties of a Binomial Experiment

1. There are n "trials" where n is determined in advance. (10 Coin flips, births, T/F questions, adults polled)

2. There are *the same two possible outcomes* on each trial, called "success" and "failure" and denoted S and F. (Heads/tails; Boy/girl; Right/wrong, Unsafe/not unsafe)

3. The *outcomes are independent* from one trial to the next. Knowledge of one does not help predict the next one. (True for all 4 examples.)

4. The probability of a "success" *remains the same* from one trial to the next, and this probability is denoted by p. The probability of a "failure" is $1-p$ for every trial.

Note that n = 10 and p = ½ for each example given.

NOTE: p = ½ is not always the case!
For example, multiple choice test with 4 choices, student is just guessing, p = 1/4.

A **binomial random variable** is defined as X = number of successes in the n trials of a binomial experiment.

<u>Two examples (one binomial, one not):</u>
Weekly quiz has 5 questions with 4 choices per question, worth 2 points each. Suppose someone is just guessing.
X = *Number of questions correct*
X **is** a binomial random variable, n = 5 and p = 1/4
Y = *Points earned for the quiz* = 2X
Y **is not** a binomial random variable (but Y/2 is).

Examples that are *not* binomial experiments:

1. A chess player plays 12 *different* opponents in a tournament, X = number of games won.

    p = Probability of win does *not* stay the same

    Condition #4 does not hold.

2. Woman decides to have children until she has one girl or 4 children, whichever comes first.
Number of "trials" is not fixed in advance (Condition #1).

3. Deal a poker hand of 5 cards, X = number of aces.
Cards are drawn *without replacement* so outcomes are NOT independent (also, p changes). (Conditions #3, #4)

Once you recognize a binomial random variable, the pdf is always given by this formula (so you don't have to rely on Chapter 7 rules each time!):

Probability of exactly *k* successes:

$$\boxed{P(X = k) = \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \quad \text{for } k = 0, 1, 2, \ldots, n.}$$

Factorial notation: $n! = 1 \times 2 \times 3 \times \ldots \times (n-1) \times (n)$

$0! = 1$, by convention.

EX: If just guessing, what is the probability of getting exactly 2 quiz questions right (out of 5 for the week)?

n = 5 ["trials" = questions], p = .25 [success prob.], k = 2

$$\boxed{P(X=2)=\frac{5!}{2!(5-2)!}(.25)^2(1-.25)^{5-2}=10(.0625)(.4219)=.2637}$$

What is the probability of getting 0 questions right?

$$\boxed{P(X=0)=\frac{5!}{0!5!}(.75)^5=(.75)^5=.2373}$$

How is the pdf formula found? Use Chapter 7 rules.
Simpler example: $n = 3$, $p = .25$, $k = 2$:

$$\mathrm{P}(X = 2) = \frac{3!}{2!(3-2)!}(.25)^2(1-.25)^{3-2} = 3(.0625)(.75) = .14$$

- Individual string of k successes and $(n - k)$ failures has probability $p^k(1\text{-}p)^{n\text{-}k}$ <u>Example</u>: P(SSF) = (.25)(.25)(.75)

- There are $\dfrac{n!}{k!(n-k)!}$ possible ways to get k successes

Example: $n = 3$, $k = 2$, could be {SSF, SFS, FSS}

$$\frac{n!}{k!(n-k)!} = \frac{3!}{2!(3-2)!} = \frac{3\times 2\times 1}{(2\times 1)(1)} = 3$$

Use computer to find binomial probabilities (pdf and cdf):

Excel – See page 278
=BINOMDIST(k,n,p,false) for the pdf
=BINOMDIST(k,n,p,true) for the cdf
(You type the equal sign then the command in any cell and it will put the requested probability in that cell.)

EX (previous slide):
=BINOMDIST(2,3,.25,false) would give .14

R Commander: See instructions linked to website.
For pdf: *Distributions* →*Discrete distributions* → *Binomial distribution* → *Binomial probabilities*
(then fill in n and p in the popup box)

Mean and standard deviation *for binomial random variables* (only!):

Mean = expected value of X = E(X) = μ = ***np***

Variance = $\sigma^2 = np(1\text{-}p)$; standard deviation = $\sqrt{np(1-p)}$

Example:

n = 10, p = 0.2

mean = (10)(0.2) = **2**

standard deviation = $\sqrt{10(.2)(.8)} = \sqrt{1.6}$ = 1.265

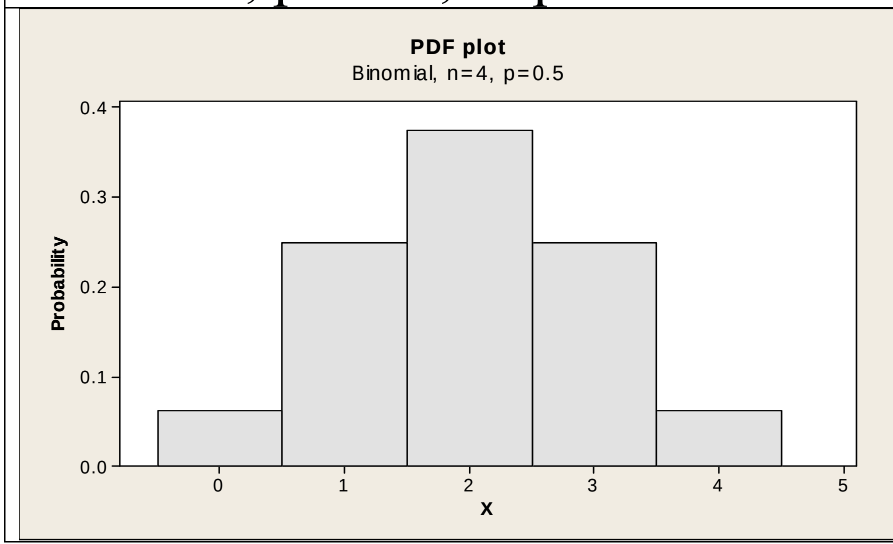
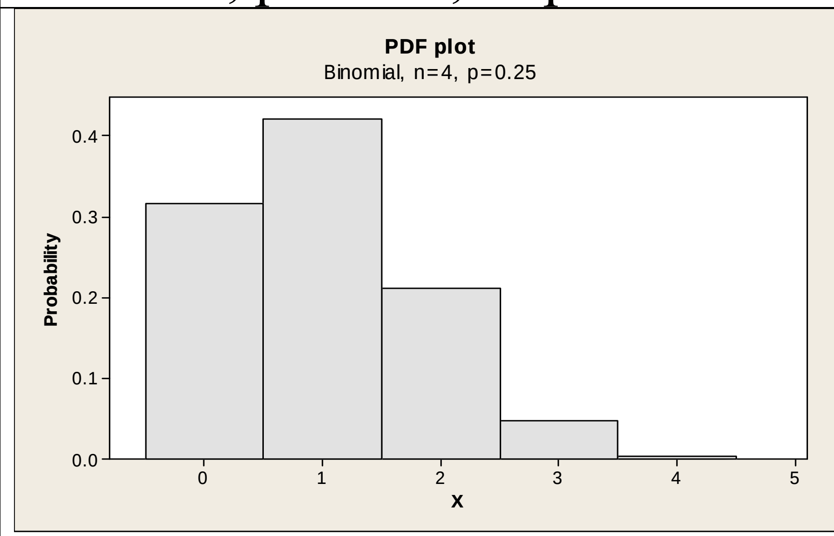
(not much use for now, but will be very useful soon)

Let's look at some pictures of binomial pdfs with different n's and p's.

Binomial pdfs, n = 4 and p = .5 (on left) or .25 (on right)

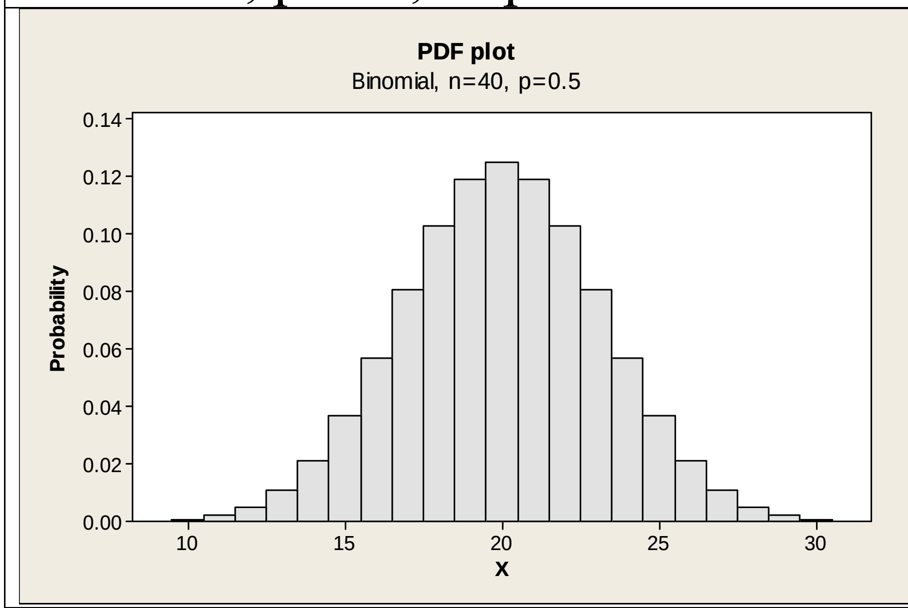
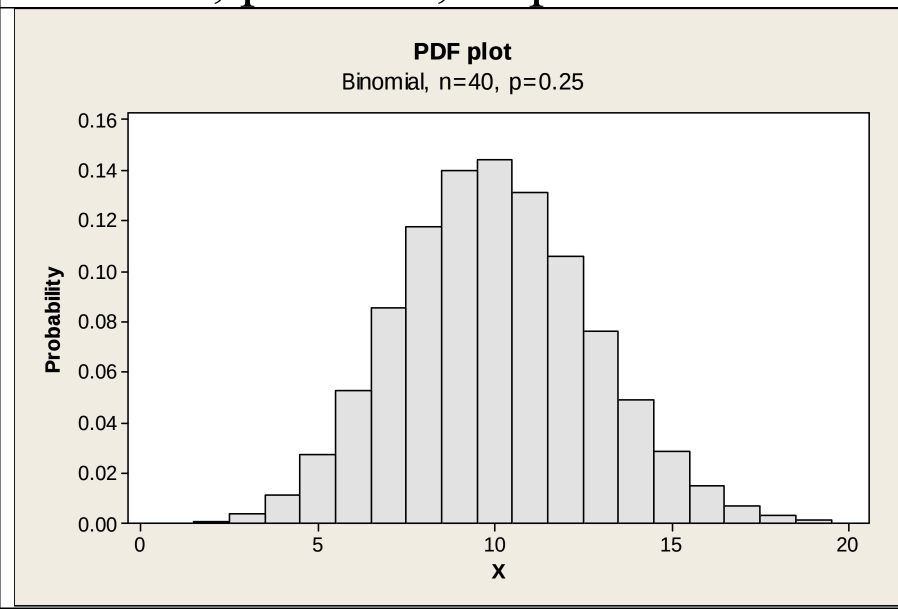
E(X) = np is (4)(.5) = 2

E(X) = np = 4(.25) = 1

n = 4, p = 0.5, Exp. value = 2	n = 4, p = 0.25, Exp. value = 1
	

Binomial pdfs, n = 40 and p = .5 (left) or .25 (right):

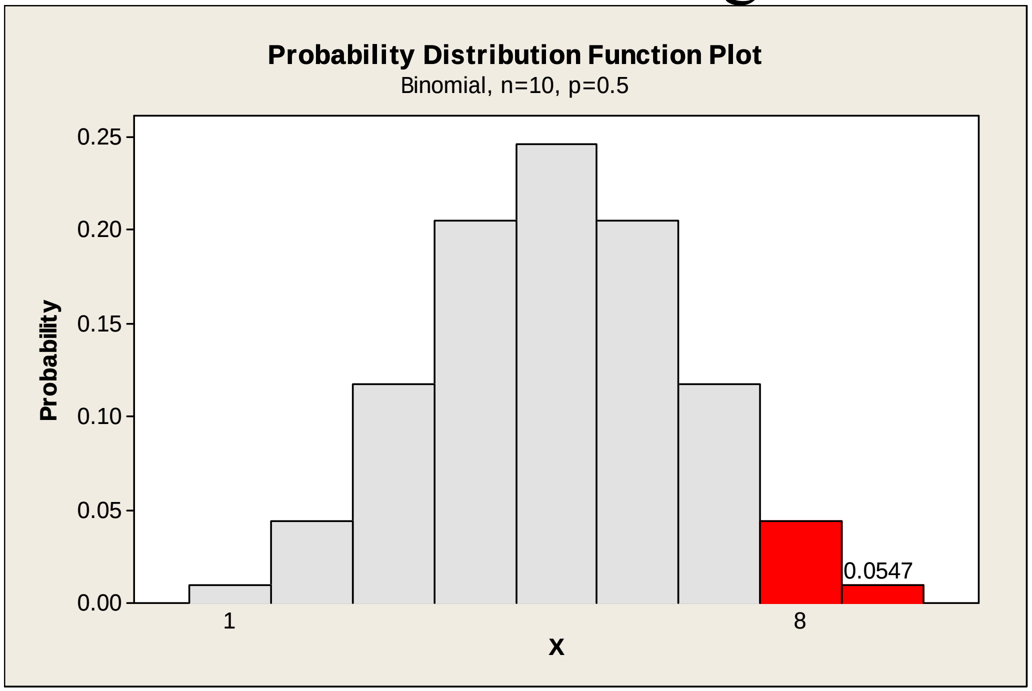
E(X) = np is (40)(.5) = 20 E(X) = np = 40(.25) = 10

n=40, p=0.5, Exp. value = 20	n=40, p = 0.25, Exp. value = 10
	

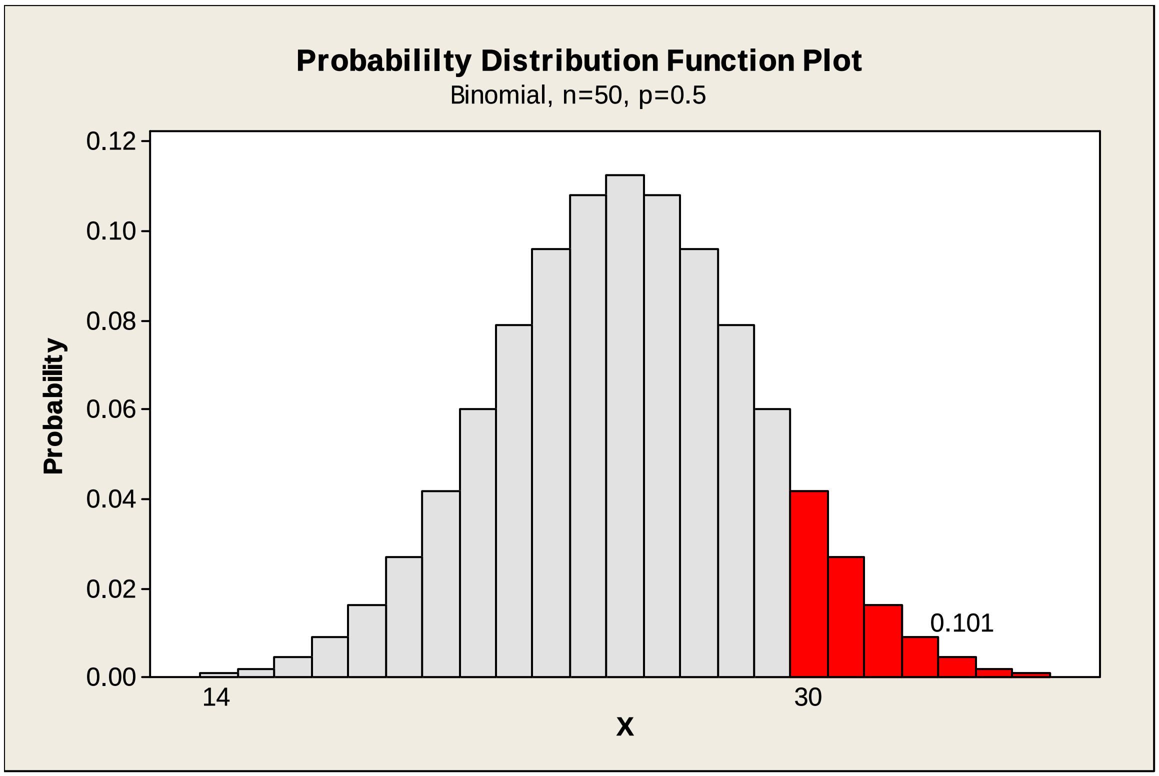
Notice how the “shape” is getting close to bell-shaped!

For binomial, CDF is often more interesting than PDF.
Ex: Test has 10 questions, pass if 80%, 8 or more, correct.

Find $P(X = 8, 9, 10) = P(X \geq 8) = 1 - P(X \leq 7) = 1 - \text{cdf}$ for $X = 7$, which is **1 −.94531 = .0547** (if just guessing)
Probability = sum of areas of rectangles for those values!



Now suppose test has 50 questions, you need 60% correct to pass, so need 30 questions correct. If just guessing, $P(X \geq 30) = 1 - P(X \leq 29) = 1 - .899 = .101$ $= P(30)+P(31)+P(32)+\ldots.+P(50)$



Ex: Political poll with n = 1000.

Suppose *true p = .48* in favor of a candidate.

X = number in poll who say they support the candidate.

X is a binomial random variable, n = 1000 and p = .48.

- n trials = 1000 people (without replacement, but for large population treat as if with replacement)
- *"success"* = support, *"failure"* = doesn't support
- Trials are *independent*, knowing how one person answered doesn't change others probabilities
- p =.48 remains fixed at for each random draw of a person to ask

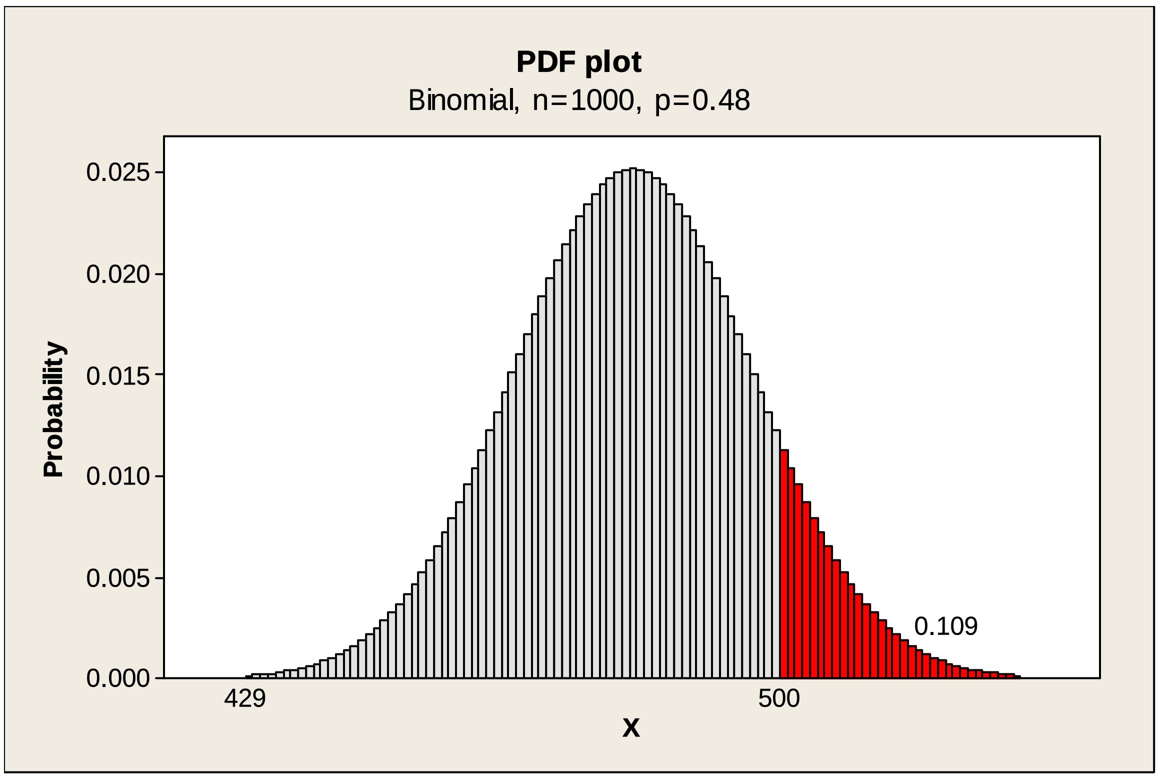
Mean = np = (1000)(.48) = **480**.

Standard dev, = $\sqrt{np(1-p)} = \sqrt{1000(.48)(.52)}$ = **15.8**

What is the probability that *at least half* of the *sample* support the candidate? (Remember only 48% of population supports him or her.)

$P(X \geq 500) = P(X = 500) + P(X = 501) + \ldots + P(X = 1000)$.

Using Excel: $1 – P(X \leq 499) = 1 – .891 = .109$.



Note what this says:
In polls of 1000 people in which 48% favor something, the poll will say *at least half favor it* with probability of just over .10 or in just over 10% of polls.

In Section 8.7, will learn how to *approximate* this using normal curve.

Binomial example you can try: Online ESP test:
http://www.gotpsi.org

Try doing 5 guesses where there are 5 choices each time.
Assuming no ESP, $n = 5$ and $p = 1/5$ or .2.
What should be expected by chance?
X = number correct, E(X) = np = (5)(1/5) = 1.
PDF is $P(X = k)$, CDF is $P(X \leq k)$
Also interesting to find $P(X \geq k)$

k	**pdf**	**cdf**	**P(X ≥ *k*)**
0	0.32768	0.32768	1.00000
1	0.40960	0.73728	0.67232
2	0.20480	0.94208	0.26272
3	0.05120	0.99328	**0.05792**
4	0.00640	0.99968	**0.00672**
5	0.00032	1.00000	**0.00032**